18.721 PSet 9

Due: Apr 19, 11:59 PM

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1. (2 points) Let X be the complement of the origin in $\mathbb{A}^2$. Compute $H^i(\mathcal{O}_X)$, and use this to give another argument that X is not affine.

Solution: We can cover X by the affine opens $\operatorname{Spec}\mathbb{C}[x, x^{-1}, y]$ and $\operatorname{Spec}\mathbb{C}[x, y, y^{-1}]$. The intersection of these two affine opens is $\operatorname{Spec}\mathbb{C}[x, x^{-1}, y, y^{-1}]$. Thus, the resulting Čech complex for $\mathcal{O}_X$ is

$$0 \to \mathbb{C}[x, x^{-1}, y] \oplus \mathbb{C}[x, y, y^{-1}] \to \mathbb{C}[x, x^{-1}, y, y^{-1}] \to 0.$$

The zeroth cohomology $H^0(\mathcal{O}_X)$ is thus important to the set of pairs $(a, b) \in \mathbb{C}[x, x^{-1}, y] \oplus \mathbb{C}[x, y, y^{-1}]$ such that the images of a and b in $\mathbb{C}[x, x^{-1}, y, y^{-1}]$ agree. This happens exactly when a and b are both equal to the same element of $\mathbb{C}[x, y]$, so $H^0(\mathcal{O}_X) \cong \mathbb{C}[x, y]$.

On the other hand, the first cohomology $H^1(\mathcal{O}_X)$ will be the quotient of $\mathbb{C}[x, x^{-1}, y, y^{-1}]$ by the image of $\mathbb{C}[x, x^{-1}, y] \oplus \mathbb{C}[x, y, y^{-1}]$. This image is generated as a vector space by the monomials $x^i y^j$ where not both i and j are negative. Thus, this quotient $H^1(\mathcal{O}_X)$ is generated by the monomials $x^i y^j$ with i and j both negative, and is thus isomorphic to $x^{-1}y^{-1}\mathbb{C}[x^{-1}, y^{-1}]$. As this is nonempty, X cannot be affine.

2. (2 points, Artin 7.10.3) Let

$$0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$$

be a complex of finite-dimensional vector spaces, and let $C_i \cong \ker(V_i \to V_{i+1}) / \operatorname{im}(V_{i-1} \to V_i)$ be the ith cohomology of this complex. Show that

$$\sum_{i=0}^{n} (-1)^i \dim V_i = \sum_{i=0}^{n} (-1)^i \dim C_i.$$

Solution: Let $U_i \subseteq V_i$ be the kernel of $V_i \to V_{i+1}$ and let $W_i \subseteq V_i$ be the image of $V_{i-1} \to V_i$. Note that for each i, we have an exact sequence

$$0 \to W_i \to U_i \to C_i \to 0,$$

so by rank-nullity, we must have $\dim C_i = \dim U_i - \dim W_i$. Thus, we have

$$\sum_{i=0}^{n} (-1)^i \dim C_i = \sum_{i=0}^{n} (-1)^i \dim U_i - \sum_{i=0}^{n} (-1)^i \dim W_i.$$

On the other hand, we also have exact sequences

$$0 \to U_i \to V_i \to W_{i+1},$$

so $\dim V_i = \dim U_i + \dim W_{i+1}$. Thus,

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \dim V_i &= \sum_{i=0}^{n} (-1)^i \dim U_i + \sum_{i=0}^{n} (-1)^i \dim W_{i+1} \\
&= \sum_{i=0}^{n} (-1)^i \dim U_i - \sum_{i=0}^{n} (-1)^i \dim W_i \\
&= \sum_{i=0}^{n} (-1)^i \dim C_i,
\end{aligned}$$

as desired.

3. Let R be a commutative ring. We say that a R-module M is projective if there is an integer n and another R-module N such that there is an isomorphism of R-modules $M \oplus N \cong R^n$.

   (a) (2 points) If R is a finite type $\mathbb{C}$-algebra, show that the quasicoherent sheaf corresponding to a projective module M as above is a vector bundle on $\operatorname{Spec} R$. (Hint: For any point p in $\operatorname{Spec} R$ corresponding to a maximal ideal m, choose a basis of M/mM and lifts m_i of the basis elements to M. Show that M becomes isomorphic to the free module generated by these elements after localization by some element of R not vanishing at p. One possible way to show this is by choosing similar elements n_i of N, and then examining when the m_i and n_i collectively generate $M \oplus N \cong R^n$.)

Solution: We follow the hint. Let p be a point of $\operatorname{Spec} R$, and assume that $M \oplus N \cong R^n$. Then $M/mM \oplus N/mN$ is isomorphic to $R^n/mR^n \cong \mathbb{C}^n$, so the dimensions a and b of M/mM and N/mN must sum to n. Choose elements $m_1, m_2, \cdots, m_a$ of M whose reductions mod m give a basis for M/mM.

Similarly, choose elements $n_1, n_2, \cdots, n_b$ of N whose reductions mod m give a basis for N/mN.
The elements m_i (resp. the elements n_i) define a map $R^a \to M$ (resp. a map $R^b \to N$.) We will show that both of these maps become isomorphisms after localizing by some element f which is not in m. It suffices to show that the sum of these maps $R^n \to M \oplus N \cong R^n$ becomes an isomorphism after some such localization. A R-module map $R^n \to R^n$ corresponds to a $n \times n$ matrix, and the map is an isomorphism if and only if the determinant of this matrix is invertible. So if we set f to be this determinant, then our maps will become isomorphisms after localization by f.
It remains to show that f is not in m (so in particular, is not zero.) The reduction of f mod m is the determinant of the matrix corresponding to

$$R/m^n \to M/mM \oplus N/mN \cong R/m^n.$$

But because how we chose our m_i and n_i, this map is an isomorphism, so the reduction of f mod m must be invertible and f cannot be in m.
The fact that M_f is isomorphic to a free module over R_f implies that the quasicoherent sheaf corresponding to M becomes free after restriction to the open $\operatorname{Spec} R_f$, which contains p (because f is not in m.) As p was arbitrary, this shows that the quasicoherent sheaf corresponding to M becomes free in an open neighborhood of every point, hence is locally free.

(b) (1 point) It is slightly difficult at the moment for us to exhibit a finite type $\mathbb{C}$-algebra with modules that are projective but not free. However, there are simple examples coming from number theory. Recall that the ring $R = \mathbb{Z}[\sqrt{-5}]$ is not a principal ideal domain because the ideal $(2, 1+\sqrt{-5})$ is not principal. Show that $(2, 1+\sqrt{-5})$ is projective but not free as a module.

Solution: This can be done conceptually using the theory of Dedekind domains. Here, we give a more computational proof.
Let I_1 denote the ideal $(2, 1+\sqrt{-5})$ and let I_2 denote the ideal $(3, 1+\sqrt{-5})$. The module $I_1 \oplus I_2 \subseteq R^2$ is tautologically generated by $(2,3), (1+\sqrt{-5}, 3), (2, 1+\sqrt{-5})$, and $(1+\sqrt{-5}, 1+\sqrt{-5})$. We claim that it is in fact generated by only $(1+\sqrt{-5}, 1+\sqrt{-5})$ and $(2,3)$. As these two elements are linearly independent over $\mathbb{Q}[\sqrt{-5}]$, this will imply that $I_1 \oplus I_2$ is free (hence I_1 is projective.)
It suffices to show that $(1+\sqrt{-5}, 3)$ and $(2, 1+\sqrt{-5})$ can be written as linear combinations (with coefficients in R) of $(2,3)$ and $(1+\sqrt{-5}, 1+\sqrt{-5})$. This follows from the identities

$$(1+\sqrt{-5}, 3) = (2-\sqrt{-5})(2,3) + (2+\sqrt{-5})(1+\sqrt{-5}, 1+\sqrt{-5})$$

and

$$(2, 1+\sqrt{-5}) = (-1+\sqrt{-5})(2,3) + (-1-\sqrt{-5})(1+\sqrt{-5}, 1+\sqrt{-5}).$$

We have thus shown that I_1 is projective. It is clearly not free, because it is not generated by one element (as it is not principal).

4. Note that the definition of Čech cohomology still makes sense for general (i.e. not necessarily quasicoherent) sheaves of abelian groups on X (with the Zariski topology). More precisely, for such a sheaf $\mathcal{F}$, we can define the ith Čech cohomology $H^i(\mathcal{F}, \{U_i\})$ of $\mathcal{F}$ with respect to a cover by opens U_i as the ith cohomology of the Čech complex

$$\prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \cap U_j) \to \prod \mathcal{F}(U_i \cap U_j \cap U_k) \cdots .$$

   (a) (1 point) Let $\mathcal{O}^*$ be the sheaf with $\mathcal{O}^*(U)$ defined to be the multiplicative group of invertible elements of $\mathcal{O}(U)$. Assume we have an open cover of X by two affine opens U_1, U_2. Compute $H^1(\mathcal{O}^*, \{U_1, U_2\})$ explicitly.

Solution: This is the first cohomology of the complex

$$0 \to \mathcal{O}^*(U_1) \oplus \mathcal{O}^*(U_2) \to \mathcal{O}^*(U_1 \cap U_2) \to 0.$$

Explicitly, we find that

$$H^1(\mathcal{O}^*, \{U_1, U_2\}) \cong \mathcal{O}^*(U_1 \cap U_2)/\mathcal{O}^*(U_1)\mathcal{O}^*(U_2).$$

   (b) (2 points) Use your description from the previous part to show that $H^1(\mathcal{O}^*, \{U_1, U_2\})$ is isomorphic to the group of line bundles on X which become trivial when restricted to both U_1 and U_2.

   You may use the following result, which is a generalization of Artin's proposition 6.4.7 (and has the same proof):

   Lemma 1. *Let S be the set of triples $\{\mathcal{L}, f_1 : \mathcal{L}|_{U_1} \to \mathcal{O}(U_1), f_2 : \mathcal{L}|_{U_2} \to \mathcal{O}(U_2)\}$ such that f_1 and f_2 are isomorphisms (and so in particular, $\mathcal{L}$ is a line bundle). Then elements of S are in bijection with isomorphisms $\mathcal{O}(U_1)|_{U_1 \cap U_2} \to \mathcal{O}(U_2)|_{U_1 \cap U_2}$. The bijection sends an element $\{\mathcal{L}, f_1, f_2\}$ to the isomorphism*

$$\mathcal{O}(U_1)|_{U_1 \cap U_2} \xrightarrow{f_1^{-1}} \mathcal{L}_{U_1 \cap U_2} \xrightarrow{f_2} \mathcal{O}(U_2)|_{U_1 \cap U_2}.$$

   In particular, if we knew there existed an affine variety X with such a line bundle (as suggested by the previous problem), this would show that the higher Čech cohomology groups do not always vanish on affine varieties if we are looking at non-quasicoherent sheaves.

   (The statement of this problem in fact still holds for an affine open cover with more than two opens, but is significantly conceptually harder.)

Solution: The lemma tells us that S is isomorphic to the group of automorphisms of $\mathcal{O}(U_1 \cap U_2)$. Each such automorphism corresponds to an invertible element of $\mathcal{O}(U_1 \cap U_2)$, so $S \cong \mathcal{O}^*(U_1 \cap U_2)$.

There is a surjection, defined by $\{\mathcal{L}, f_1, f_2\} \mapsto \mathcal{L}$, from S to the group of line bundles on X which become trivial when restricted to both U_1 and U_2. The kernel of this surjection comes from elements of S with $\mathcal{L} \cong \mathcal{O}$. In this case, choices of f_1 and f_2 corresponds to choosing elements of $\mathcal{O}^*(U_1)$ and $\mathcal{O}^*(U_2)$, so the kernel is generated by these groups, and so the group of line bundles on X which becomes trivial when restricted to both U_1 and U_2 is isomorphic to $\mathcal{O}^*(U_1 \cap U_2)/\mathcal{O}^*(U_1)\mathcal{O}^*(U_2)$, as desired.

5. (1 point) Try to imagine how you might have come up with the concept of sheaf cohomology. Give as plausible an explanation as you can think of.

Solution: Many possible answers.