Necessity as Truth in All Possible Worlds

What redeems modal logic, making it something more that an empty exercise in writing down arbitrary rules and then following them, is possible-world semantics. The key idea behind possible-world semantics is this: A sentence is necessary if and only if it is true in all possible worlds. Modal logic is adaptable, and we might want to allow variations on this key idea for various purposes, but right now, let us examine the key idea.

We begin by setting up a formalism, extending the sentential calculus to the modal sentential calculus by introducing a new operator. The *MSC sentences* constitute the smallest class of expressions that:

> the atomic sentences;
> contains $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$ whenever it contains φ and ψ; and
> contains $\sim \varphi$ and $\Box\, \varphi$ whenever it contains φ and ψ.

We're treating $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$, and $\Diamond\, \psi$ as defined by $(\sim \varphi \vee \psi)$, $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, and $\sim \Box \sim \varphi$.

We define a *simple Kripke model* to be an ordered triple <W,I,@>, where W is a set of things we'll be calling "worlds"; I is a function that assigns to each pair <α,w>, where α is an atomic sentence and w is a world, a value either True or False; and @ is a member of W we'll call "the actual world." What possible worlds are really is a hard question we'll take up when we get to the modal predicate calculus. For now, all we require of W is that it be a nonempty set. We're doing simple Kripke models. Complicated Kripke models will come next week.

Let w be a world in a simple Kripke model. We specify what it is for a sentence to be true in w in the model by induction.

> An atomic sentence α is true in w iff $I(\alpha,w) =$ True.
>
> A disjunction is true in w iff one or both disjuncts are true in w.
>
> A conjunction is true in w iff both conjuncts are true in w
>
> A negation is true in w if its negatum isn't true in w.
>
> $\Box\, \varphi$ is true in w iff φ is true in every member of W.

A sentence is true in the model <W,I,@> iff it's true in @.

For any world w, declaring a sentence true iff it's true in w yields a NTA, where NTA is defined as it was for SC, treating sentences that begin with "□" like atomic sentences. It follows that every tautology (that is, every sentence assigned "True" by every NTA) is true in every world in every model. We'll use "(Taut") to refer to the set of tautologies.

If $\Box(\varphi \rightarrow \psi)$ and $\Box\,\varphi$ are both true in a world w, both $(\varphi \rightarrow \psi)$ and φ are true in every world. So ψ is true in every world, so $\Box\,\psi$ is true in w. It follows that every sentence of the following form is true in every world in every model:

(K) $(\Box(\varphi \rightarrow \psi) \rightarrow (\Box\,\varphi \rightarrow \Box\,\psi))$.

If $\Box\,\varphi$ is true in w, φ is true in every world, so, in particular, φ is true in w. So sentences of the following form are true in every world in every model:

(T) $(\Box\,\varphi \rightarrow \varphi)$/

If $\Box\,\varphi$ is rrue in w, φ is true in every world. For any world v, since φ is true in every world, $\Box\,\varphi$ is true in v. Since $\Box\,\varphi$ is true in every world, $\Box\,\Box\,\varphi$ is true in w. It follows that instances of the following schema are true at every world in every model:

(4) $(\Box\,\varphi \rightarrow \Box\,\Box\,\varphi)$.

The axiom is called "(4)" because it was number four in C. I. Lewis's list of axioms. Don't try to make sense of the names of the formulas.

If $\Diamond\,\varphi$ is true in w, there is a world in which φ is true. For any world v, since there is a world in which φ is true, $\Diamond\,\varphi$ is true in v. Since $\Diamond\,\varphi$ is true in every world, $\Box\,\Diamond\,\varphi$ is true in w. Thus instances of the following schema are true in every world in every model:

(5) $(\Diamond\,\varphi \rightarrow \Box\,\Diamond\,\varphi)$.

If $(\varphi \rightarrow \psi)$ and φ are both true in every world in every model, then ψ is true in every world in every model. So the set of sentences true in every world in every model is closed under modus ponens:

(MP) From $(\varphi \rightarrow \psi)$ and φ, you may infer ψ.

If φ is true in every world in every model, $\Box\,\varphi$ is true at every world in every model. Thus the set of sentences true in every world in every model is closed under Necessitation:

(Nec) From φ, you may infer $\Box\,\varphi$.

S5 is the smallest collection of sentences that includes all the sentences of the forms (Taut), (K), (T), (4), and (5) and in closed under (MP) and (Nec).

We have proved the left-to-right direction of the following:

> **Theorem.** A sentence is in S5 and only if it is true in every world in every simple model.

To get the other direction, we need to show that, if we have a sentence that isn't in S5, there is a simple Kripke model in which it's false. The plan is to take our possible worlds to be complete stories. However, we don't want to require that every complete story is a possible world. Thinking in terms of logical necessity, we want to treat a sentence as necessaary iff it's made true by the rules of the language. Every tautology is made true by the rules that govern the sentential connectives, so every tautology is necessary. However, a sentence might be made true by rules of the language to go beyond the rules that govern the sentential connectives. For example, "Every square is a rhombus," "Anything that's scarlet is red," and "Nothing is both blue all over and yellow all over." These sentences aren't tautologies, so there are complete stories in which they are false, but they are made true by the rules of the language. So we wouldn't want to count a complete story that includes them as a possible world. So our set W will be a nonempty set of complete stories, but it need not be the entire set of complete stories.

Suppose that χ isn't in S5. Because S5 includes (Taut) and is closed under (MP), χ isn'a tautological consequence of any finite subset of S5. By the Compactness Theorem for SC, χ isn't a tautological consequence of S5, and so there is a complete story @ that includes S5 and excludes χ. Let W be the set of complete stories that include all the sentences ζ with $\Box\zeta \in$ @. Because of axiom schema (T), @ $\in$ W. Note that, whenever a sentence ζ is in S5, $\Box\zeta$ is in S5, and so in @, so that ζ in in every member of W. Thus every member of W includes S5. Define an interpretation I by saying that $I(\alpha,w)$ = True iff $\alpha \in w$. We want to show that, for any world w and sentence θ, θ is true in w iff θ is an element of w. This will tell us that χ is false in <W,I,@>.

The proof is by induction on the complexity of formulas, treating the complexity of $\Box\theta$ as 1 + the complexity of θ. There are five cases; the first four are easy:

Case 1. θ atomic. This is how I was defined.

Case 2. θ is a disjunction. The θ is true in w iff one or both disjuncts are true in w iff one or both disjuncts are elements of w iff the disjunction is an element of w.

Cases 3-4. θ is a conjunction or negation. Similar.

Case 5. θ has the form $\Box\varphi$. If $\Box\varphi$ is true in w, φ is true in every world. By inductive hypothesis, φ is an element of every world. Hence φ is an element of every complete story that includes all the sentences ζ with $\Box\zeta$ in @. This means, by Lindenbaum's Lemma that there are sentences $\zeta_1, \zeta_2, ..., \zeta_n$ with each $\Box\zeta_i$ in @, such that $(\zeta_1 \to (\zeta_2 \to ... (\zeta_n \to \varphi)...))$ is a tautology. Since it's a tautology, it's in S5, and so by (Nec), the result of putting "$\Box$" in front of it is in S5. Using (K) multiple times, we see that $(\Box\zeta_1 \to (\Box\zeta_2 \to ... (\Box\zeta_n \to \Box\varphi)...))$ is in S5, and so in @. So $\Box\varphi$ is in @. Since $(\Box\varphi \to \Box\Box\varphi)$ is in S5 by (4), and so in @, $\Box\Box\varphi$ is in @, so $\Box\varphi$ is in w.

Now for the other direction, suppose that $\Box\varphi$ isn't true in w. Then there is a world v in which φ isn't true. By inductive hypothesis, $\varphi \notin v$. So $\Box\varphi$ isn't in @. So $\sim\Box\varphi$ is in @. Using

(5), we can derive $(\sim \Box \varphi \rightarrow \Box \sim \Box \varphi)$, and so establish that $\Box \sim \Box \varphi$ is in @. Consequently $\sim \Box \varphi$ is in w, and $\Box \varphi$ isn't in w. ⊠

Short cut rules

An *S5 derivation* is a finite sequence of sentences, each of which either has one of the forms (Taut), (K), (T), (4), or (5) or follows from earlier members of the sequence by (MP) or (Nec). A sentence is in S5 if and only if it appears within some S5 derivation.

As an example, let's give a derivation of "$(\Box(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q))$":

1.	$((P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P))$	(Taut)
2.	$\Box((P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P))$	(Nec), 1
3.	$(\Box((P \rightarrow Q) \rightarrow (\neg Q \rightarrow \sim P)) \rightarrow (\Box(P \rightarrow Q) \rightarrow \Box(\sim Q \rightarrow \sim P)))$	(K)
4.	$(\Box(P \rightarrow Q) \rightarrow \Box(\sim Q \rightarrow \sim P))$	(MP), 2, 3
5.	$(\Box(\sim Q \rightarrow \sim P) \rightarrow (\Box \sim Q \rightarrow \Box \sim P))$	(K)
6.	$(\Box \sim Q \rightarrow \Box \sim P) \rightarrow (\Diamond P \rightarrow \Diamond Q))$	(Taut), def. of "◊"
7	$((4) \rightarrow ((5) \rightarrow ((6) \rightarrow ((\Box(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q)))))$	(Taut)
8.	$((5) \rightarrow ((6) \rightarrow (\Box(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q))))$	(MP), 4, 7
9.	$((6) \rightarrow (\Box(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q))\}$	(MP) 5. 7
10.	$(\Box(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q))$	(MP) 6, 9

This is as simple as derivations come, yet writing it out is unpleasantly convoluted and time-consuming. We won't be spending a lot of time doing derivations, but for when we do, let's introduce some short cut rules that will simplify. These are derived rules. Anything you can prove using the rules you can prove without it

> **(K).** This is the rule (K) which is distinct from the axioms schema (K). If you have derived $\Box(\varphi \rightarrow \psi)$, you may derive $(\Box \varphi \rightarrow \Box \psi)$.

We could get the same effect by writing an instance of (K) and employing (MP).

> **(TC)** You may write down a sentence that is either a tautology or a tautological consequence of earlier sentences.

The new rule sidesteps writing an instance if (Taut) followed by one or more applications of (MP). A rule that told you you could write down any sentence that in S5 wouldn't be useful, at least as of now, because we don't have any way of telling when the rule has been applied. (TC) isn't susceptible to the same complaint, because we can us a truth table to verify that a purported application is correct. Later on, we'll get a test for validity in S5.)

For the next rule, we need a definition. A *substitution* is a function s taking sentences to sentences that meets the following conditions:

$s(\varphi \vee \psi) = (s(\varphi) \vee s(\psi))$.
$s(\varphi \wedge \psi) = (s(\varphi) \wedge s(\psi))$.
$s(\sim \varphi) = \sim s(\varphi)$.
$s(\Box\, \varphi) = \Box\, s(\varphi)$.

$s(\varphi)$ is said to be a *substitution instance* of φ. Any function that takes atomic sentences to sentences can be extended to a substitution in a unique way.

(Subs) For s a substitution, if you've derived φ, you may write $s(\varphi)$.

Noting the a substitution instance of an axiom is an axiom, we can perform the substitution uniformly on every line of a derivation, getting s derivation of $s(\varphi)$.

(Equiv) Let s_φ be the substituion that takes the atomic sentence "R" to φ and leaves every other atomic sentence unmoved. Let s_ψ be the substitution that takes "R to ψ and leaves every other atomic sentence alone. For any sentence θ, if you have derived $(\varphi \leftrightarrow \psi)$, you may derive $(s_\varphi(\theta) \leftrightarrow s_\psi(\theta))$ and also $(s_\psi(\theta) \leftrightarrow s_\varphi(\theta))$.

We'll prove that $(s_\varphi(\theta) \leftrightarrow s_\psi(\theta))$ is derivable by induction on the complexity of θ. That $(s_\psi(\theta) \leftrightarrow s_\varphi(\theta))$ is derivable will follow by (TC). There are six cases:

Case 1. θ = "R". Then $s_\varphi(\theta) = \varphi$ and $s_\psi(\theta) = \psi$.

Case 2. θ is an atomic sentence other than "R." Then $s_\varphi(\theta) = s_\psi(\theta) = \theta$.

Case 3. θ is a disjunction, say $(\mu \vee \nu)$. Then by inductive hypothesis, $((s_\varphi(\mu) \leftrightarrow s_\psi(\mu))$ and $s_\varphi(\nu) \leftrightarrow s_\psi(\nu))$ are both derivable. A derivation of $((s_\varphi(\theta) \leftrightarrow s_\psi(\theta))$ is obtained by (TC)

Cases 4 and 5. θ is a conjunction or a negation. Similar.

Case 6. θ has the form $\Box\,\mu$. By inductive hypothesis, there is a derivation of $(s_\varphi(\mu) \leftrightarrow s_\psi(\nu))$, from which we derive $(s_\varphi(\mu) \rightarrow s_\psi(\nu))$ and $(s_\psi(\mu) \rightarrow s_\varphi(\nu))$ by (TC). (Nec) gives us $\Box(s_\varphi(\mu) \rightarrow s_\psi(\nu))$ and $\Box(s_\psi(\mu) \rightarrow s_\varphi(\nu))$, from which we get $(\Box s_\varphi(\mu) \rightarrow \Box s_\psi(\nu))$ and $(\Box s_\psi(\mu) \rightarrow \Box s_\varphi(\nu))$ by rule (K). (TC) yields $(\Box s_\varphi(\mu) \leftrightarrow \Box s_\psi(\nu))$, which is the same as $(s_\varphi(\Box\mu) \rightarrow s_\psi(\Box\nu))$. ⊠

Let's do a couple of examples. First, a derivation of $(\Diamond\varphi \leftrightarrow \Diamond\,\Diamond\,\varphi)$, which is an abbreviation of $(\sim \Box \sim P \leftrightarrow \sim \Box \sim \sim \Box \sim P)$:

1.	$(\Box \sim P \rightarrow \Box\,\Box \sim P)$	(4)
2.	$(\Box\,\Box \sim P \rightarrow \Box \sim P)$	(T)
3.	$(\sim \Box \sim P \leftrightarrow \sim \Box\,\Box\, P)$	(TC) 1, 2
4.	$(\Box\, P \leftrightarrow \sim \sim \Box\, P)$	(Taut)
5.	$(\sim \Box\,\Box\, P \leftrightarrow \sim \Box \sim \sim \Box\, P)$	(Equiv) 4

6.	$(\sim \Box \sim P \leftrightarrow \sim \Box \sim \sim \Box \sim P)$	(TC) 3, 5

Now we prove the dual of (5), $(\Diamond \Box \varphi \rightarrow \Box \varphi)$ (we'll explain what "dual" means in a minute):

1.	$(\Diamond \sim \varphi \rightarrow \Box \Diamond \sim \varphi)$	(5)
2.	$(\sim \Box \sim \sim \varphi \rightarrow \Box \sim \Box \sim \sim \varphi)$	Unabbreviated form of (1)
3.	$(\sim \Box \sim \Box \sim \sim \varphi \rightarrow \Box \sim \sim \varphi)$	TC 2
4.	$(\Diamond \Box \sim \sim \varphi \rightarrow \Box \sim \sim \varphi)$	Abbreviated form of 3
5.	$(\sim \sim \varphi \leftrightarrow \varphi)$	(Taut)
6.	$(\Box \sim \sim \varphi \leftrightarrow \Box \varphi)$	(Equiv) 5
7.	$(\Diamond \Box \sim \sim \varphi \leftrightarrow \Diamond \Box \varphi)$	(Equiv) 5
8.	$(\Diamond \Box \varphi \rightarrow \Box \varphi)$	(TC) 4, 6, 7

Duality

Let θ be a formula formed from atomic sentences by means of the operators "$\vee$," "$\wedge$," "$\sim$," "$\Box$," and "$\Diamond$"; "$\rightarrow$" and "$\leftrightarrow$" are forbidden. Let Dual(θ) be formed from θ by exchanging "$\vee$"s and "$\wedge$"s and exchanging "$\Box$"s and "$\Diamond$"s. Let s_{Neg} be the substitution that replaces each atomic sentences with its negation. We have the following:

Duality Theorem. $(\text{Dual}(\theta) \leftrightarrow \sim s_{Neg}(\theta))$ is in S5.

Proof: By induction on the complexity of θ. There are six cases:

Case 1. θ is atomc. Dual(θ) = θ, which is provably (in S5) equivalent of $\sim \sim \theta = \sim s_{Neg}(\theta)$

Case 2. θ is a disjunction, say $(\varphi \vee \psi)$. $\text{Dual}(\varphi \vee \psi) = (\text{Dual}(\varphi) \wedge \text{Dual}(\psi))$. By inductive hypothesis, this is provably equivalent to $(\sim s_{Neg}(\varphi) \wedge \sim s_{Neg}(\psi)$, which is provably equivalent to $\sim (s_{Neg}(\varphi) \vee s_{Neg}(\psi)) = \sim s_{Neg}(\varphi \vee \psi)$.

Cases 3-4. θ is a conjunction or a negation. Similar.

Case 5. θ has the form $\Box \varphi$. $\text{Dual}(\Box \varphi) = \Diamond \text{Dual}(\varphi)$. By inductive hypothesis and (Equiv), this is provably equivalent to $\Diamond \sim s_{Neg}(\varphi)$, which is provably equivalent to $\sim \Box s_{Neg}(\varphi) = \sim s_{Neg}(\Box \varphi)$.

Case 6. θ has the form $\Diamond \varphi$. Similar. ⊠

The Duality Theorem, combined with (Subs) and (TC), gives us a rule:

> (Dual) If you have derived $(\theta \rightarrow \chi)$, you may derive $(\text{Dual}(\chi) \rightarrow \text{Dual}(\theta))$. If you have derived $(\theta \leftrightarrow \chi)$, you may derive $(\text{Dual}(\theta) \leftrightarrow \text{Dual}(\chi))$.

Modal Operators as Quantifiers

We can think of the modal operators as quantifiers. “Necessarily φ” means, “in every world, φ.” “Possibly φ” means, “in at least one world, φ.” We can make this correlation more precise by thinking about quantification in the monadic predicate calculus, the restricted version of the predicate calculus that contains only one-place predicates and that eliminates variables other that “x.” (Disallowing the other variables is no real cost. Every sentence (formula without free variables) that you can form by allowing the other variables is logically equivalent to a sentence containing only “x.”)

Given a language for the MSC, form a language for the monadic predicate calculus by introducing an atomic formula “Px” for each atomic MSC sentence “P.” Translate the MSC sentences as formulas for the monadic predicate calculus by settting t(“P”) = “Px,” $t(\varphi \vee \psi) = (t(\varphi) \vee t(\psi))$, $t(\varphi \wedge \psi) = (t(\varphi) \wedge t(\psi))$, $t(\sim \varphi = \sim t(\varphi)$, and $t(\Box\, \varphi) = (\forall x)t(\varphi)$. Every formula of our language for the monadic predicate calculus is obtained in this way.

Given a model <W,I,@> for MSC, form a model $\mathfrak{W}$ for the monadic calculus by letting the domain of the model be W, and stipulating that an element w of W is to satisfy Px iff I(P,w) = True. Every model of our language for the monadic predicate calculus can be obtained in this way.

For any world w and MSC sentence φ, we have φ is true in w in the model <W,I,@> if and only if w satisfies $t(\varphi)$ in $\mathfrak{W}$. Thus φ is in S5 if and only if $t((\forall x)\varphi)$ is logically valid.

Nested Modal Operators are Otiose

The language for MSC permits nested modal operators, sentences prefixed by “□’ that contain other sentences prefixed by “□,” which in turn contain further sentences prefixed by “□.” For some of the systems we’ll look at later, the nested “□”s serve useful purposes. Not so for S5. For each sentence containing “□”s, we can push the “□”s inside so as to be an S5-equivalent sentence with no nesting.

We want to prove this by induction on the complexity of sentences. What we’ll need to show is that, if η is a sentence with no nesting, $\Box\, \eta$ is S5-equivalent to a sentence with no nesting. We can assume η is in conjunctive normal form. Using the fact that $\Box(\mu_1 \wedge \mu_2 \wedge ... \wedge \mu_p)$ is S5-equivalent to $(\Box\, \mu_1 \wedge \Box\, \mu_2 \wedge ... \Box\, \mu_p)$, it will be enough to show that each of the sentences obtained by placing a “□” in front of one of the conjuncts of η is S5-equivalent to a sentence with no nesting. The conjuncts of η are disjunctions of sentences of the following forms:

atomic sentences;
negations of atomic sentences;
sentences obtained by placing a “□” in front of an SC sentence; and
sentences obtained by placing “~ □” in front of an SC sentence.

Rearranging these components, we are left with a sentence of the following form:

$$\alpha \vee \Box\,\beta_1 \vee \Box\,\beta_2 \vee \ldots \vee \Box\,\beta_m \vee \sim\Box\,\gamma_1 \vee \sim\Box\,\gamma_2 \vee \ldots \vee \sim\Box\,\gamma_n,$$

where α, the β_is, and the γ_js are SC sentences.

We need to show that the result of putting a "□" on front of this sentence is S5-equivalent to a sentence with no nesting of modal operators. In fact,

① $$\Box(\alpha \vee \Box\,\beta_1 \vee \Box\,\beta_2 \vee \ldots \vee \Box\,\beta_m \vee \sim\Box\,\gamma_1 \vee \sim\Box\,\gamma_2 \vee \ldots \vee \sim\Box\,\gamma_n)$$

is S5-equivalent to

② $$\Box\alpha \vee \Box\,\beta_1 \vee \Box\,\beta_2 \vee \ldots \vee \Box\,\beta_m \vee \sim\Box\,\gamma_1 \vee \sim\Box\,\gamma_2 \vee \ldots \vee \sim\Box\,\gamma_n.$$

① is provably equivalent to this:

$$\Box(\sim\Box\,\beta_1 \rightarrow (\sim\Box\,\beta_2 \rightarrow \ldots (\sim\Box\,\beta_m \rightarrow (\Box\,\gamma_1 \rightarrow (\Box\,\gamma_2 \rightarrow \ldots (\Box\,\gamma_n \rightarrow \alpha)\ldots)))\ldots)).$$

Using (K) m+n times to push the "□" through, this implies:

$$\Box\sim\Box\,\beta_1 \rightarrow (\Box\sim\Box\,\beta_2 \rightarrow \ldots (\Box\sim\Box\beta_m \rightarrow (\Box\Box\,\gamma_1 \rightarrow (\Box\Box\,\gamma_2 \rightarrow \ldots (\Box\Box\,\gamma_n \rightarrow \Box\alpha)\ldots)))\ldots)).$$

Since $(\Box\sim\Box\,\beta_i \leftrightarrow \sim\Box\,\beta_i)$ and $(\Box\,\Box\,\gamma_j \leftrightarrow \Box\,\gamma_j)$ are in S5, this is S5-equivalent to;

$$\sim\Box\,\beta_1 \rightarrow (\sim\Box\,\beta_2 \rightarrow \ldots (\sim\Box\,\beta_m \rightarrow (\Box\,\gamma_1 \rightarrow (\Box\,\gamma_2 \rightarrow \ldots (\Box\,\gamma_m \rightarrow \Box\,\alpha)\ldots)))\ldots)),$$

which is tautologically equivalent to ①.

Now we need to show that ② S5-implies ①. α implies $(\alpha \vee \Box\,\beta_1 \vee \Box\,\beta_2 \vee \ldots \vee \Box\,\beta_m \vee \sim\Box\,\gamma_1 \vee \sim\Box\,\gamma_2 \vee \ldots \vee \sim\Box\,\gamma_n)$, so $\Box\,\alpha$ S5-implies ①. $\Box\,\beta_i$ implies $(\alpha \vee \Box\,\beta_1 \vee \Box\,\beta_2 \vee \ldots \vee \Box\,\beta_m \vee \sim\Box\,\gamma_1 \vee \sim\Box\,\gamma_2 \vee \ldots \vee \sim\Box\,\gamma_n)$, so $\Box\,\beta_i$ S5-implies $\Box\,\Box\,\beta_i$, which S5-implies ①. $\sim\Box\,\gamma_j$ implies $(\alpha \vee \Box\,\beta_1 \vee \Box\,\beta_2 \vee \ldots \vee \Box\,\beta_m \vee \sim\Box\,\gamma_1 \vee \sim\Box\,\gamma_2 \vee \ldots \vee \sim\Box\,\gamma_n)$, so $\sim\Box\,\gamma_j$ S5-entails $\Box\sim\Box\,\gamma_j$, which S5-entails ①. So each of the disjuncts of ② S5-entails ①. ⊠

Normal Form

We want to develop a modal counterpart to disjunctive normal form. For now, let's restrict our attention to a language with n atomic sentences, $\alpha_1, \alpha_2, \ldots, \alpha_n$. For that language, there are 2^n state descriptions. Within any model, the worlds are parceled out into 2^n cells, some of which might be empty, according to which state descriptions they satisfy.

A *model description* is a conjunction of 2^n conjuncts, one of which is a state description, and the rest of which are obtained by prefixing either "$\Diamond$" or "$\sim \Diamond$" to each of the remaining state description. For each model, there is a unique model description that's true in the model. It gives us the state description of the actual world, and it also tells us which state descriptions are satisified in the remaining worlds.

Given a model description μ, there will be a canonical model that satisfies μ. The worlds will be state descriptions. The actual world will be the state description that appear as a conjunct of μ. The other worlds will be the state descriptions σ for which $\Diamond\ \sigma$ is a conjunct. I(α,w) will be assigned the value True iff α is one of the conjuncts of μ

Assuming we have settled on a way of ordering the conjuncts, there will be $2^{2^n + n - 1}$ model descriptions. They will partition the models into 2^{2^n+n-1} nonempty categories.

If two models <W,I,@> and <W*,I*,@*> have the same model description, then for each world in one of the models there will be one or more worlds in the other model that satisfy the same state description. A straightforward induction on the complexity of sentences shows that, if w $\in$ W and w* $\in$ W* satisfy the same state description, then w satifies a sentence in <W,I,@> iff w* satisfies it in <W*,I*,@*> . This holds, in particular, for @ and @*, so all the same sentences are true in the two models.

This implies that a model description is complete, that is, for any sentence φ, the model description S5-entails either φ or $\sim \varphi$. If not then there would be a model <W,I,@> of the model description in which φ is false and a model <W*,I*,@*> of the model description in which φ is true, and that can't happen. Since the model description is complete, any sentence that is consistent with the model description is entailed by it.

For ψ a sentence, define the *normal form* of ψ to be the disjunction of all the model descriptions that entail ψ. So the normal form of ψ entails ψ. If <W,I,@> is a model of ψ, the model description of <W,I,@> is consistent with ψ. So the model description of <W,I,@> entails ψ. So the model description of <W,I,@> is one of the disjuncts of the normal form of ψ, hence the normal form is true in <W,I,@ >. Thus ψ entails it normal form, so that a sentence and its normal form are S5-equivalent.

We can show that a sentence is in S5 by writing out an S5 derivation. How do we show a sentence isn't in S5? Failure to derive it doesn't show that it isn't derivable, only that it hasn't been derived yet. We have an answer. If a sentence isn't in S5, we can exhibit a canonical model in which it's false.

We have a procedure for showing that a sentence is in S5, namely, to derive it. We have a procedure for showing that a sentence isn't in S5, namely, to exhibit an canonical model in which it is false. Put the two procedures together, and we get an decision procedure, an algorithm for testing whether a sentence is in S5brot