Logical Necessity and S5

Strict Logical Necessity

We distinguish two kinds of truths: truths that report matters of fact, which obtain because of what the world is like, and truth that obtain because of the rules of language. We refer to the latter as "logically necessary." The terminology here is not uniform. Quine distinguishes truth on account of the rules of language, which he calls "analyticity," from the more restrictive notion he calls "logical necessity," which is truth on account of the rules that govern the logical connectives; he is deeply suspicious of both notions.[1] For the most part, we'll follow the more relaxed usage, but let us briefly examine the more restrictive notion that identifies logical necessity with truth on account of the rules of the modal sentential calculus.

A MSC sentence containing no modal operators is *strictly logically necessary* iff it's tautological. On any plausible way of thinking about necessity, if φ is tautological, $\Box\,\varphi$ is true. What's special about strict logical necessity is the converse thesis that, if φ is a SC sentence that isn't tautological, $\Box\,\varphi$ is false.

Any sentence that's tautological is true. If a conditional and its antecedent are both tautological, the consequent is tautological. If $\Box\,\varphi$ is true, it's made true by the rules of the modal sentential calculus, so $\Box\,\Box\,\varphi$ is true. If $\Box\,\varphi$ is untrue, the rules of the modal sentential calculus make it untrue, so $\Box \sim \Box\,\varphi$ is true. These considerations assure us that every sentence in S5 is strictly logically necessary. There are other strictly logically necessary sentences that aren't in S5, notably $\Diamond\,\theta$ for every consistent SC sentence θ.

Let's pause to set up some notation. Let's begin by enumerating the atomic sentences without repetitions as $\alpha_1, \alpha_2, \alpha_3, \ldots$. A *state description* for $\alpha_1, \ldots, \alpha_k$ is a conjunction $\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k$, where β_i is either α_i or $\sim \alpha_i$. This notation is not unambiguous. "$\wedge$" is a binary connective, so there are lots of different sentences that can be obtained by conjoining $\beta_1, \beta_2, \ldots, \beta_n$, with each β_i before β_{i+1}, by making different choices for which two sentences to conjoin next. We don't care about the differences among these conjunctions, so let's pretend that we have settled on some system for grouping the conjuncts. In general, let us allow ourselves to talk about conjunctions and disjunctions of finite sets of sentences, without fretting about order and grouping. Where "$\top$" abbreviates some selected tautology, say $(\alpha_1 \vee \sim \alpha_1)$, and "$\bot$" abbreviates $(\alpha_1 \wedge \sim \alpha_1)$, let's take the conjunction of the empty set of sentences to be "$\top$," and the disjunction of the empty set of sentences is "$\bot$." This will let us simplify the Disjunctive Normal Form Theorem to the statement that every SC sentence is logically equivalent to a unique disjunction of state descriptions. We no longer have to make a special provision for contradictions.

Strict S5 is the smallest collection of sentences that contains all tautologies; all instances of schemata (T), (4), and (5); and all sentences of the form $\Diamond\,\theta$, for θ a state description, for some

[1] "Two Dogmas of Empiricism," *Philosophical Review* 60 (1951): 20-43. Reprinted in *From a Logical Point of View*, 2nd ed. (Harvard University Press, 1980), pp., pp. 20-46.

initial segment of α_1, α_2,...; and is closed under (MP) and (Nec). Of the different modal systems we'll study, Strict S5 is one of very few that aren't closed under substitution. We've seen that every strictly logically necessary MSC sentence is in Strict S5. Strict S5 describes the modal logic of logical necessity. In fact, it completely describes it, deciding the truth or falsity of every modal sentence.

Suppose that χ is a sentence whose atomic sentences are among α_1,..., α_k. If χ holds in every world in the submodel of the canonical Kripke model consisting of worlds in which all the $\Diamond\sigma$s hold, $\Box\chi$ is an element of Strict S5. If χ holds in some world in the submodel of the canonical Kripke model consisting of worlds in which all the $\Diamond\sigma$s hold, $\sim \Box \chi$ is an element of Strict S5. It follows that, for any sentence that's modalized, so that every atomic sentence that appears in it occurs within the scope of a "$\Box$," either the sentences or its negation is in Strict S5.

The Logic of Logical Necessity

To ask in isolation whether the atomic sentence "P" is true isn't a sensible question. "P" is just a letter. It doesn't mean anything, and we can only ask sensibly about the truth of expressions that are meaningful. Before we can reasonably ask about the truth of sentences of the formal language, we have to interpret it.

The most direct way to interpret a formal language is to give a method for translating it into a language we already understand. The standard way to interpret the language of the modal sentential calculus is to associate an English sentence with each atomic sentence, then to go on to translate "$\vee$" as "or" and "$\wedge$" as "and." The smooth way to translate "$\sim$" is to insert "not," "none," "no," or "never" at the appropriate place in the sentence that translates the negatum, but to specify the appropriate place will not be easy. A ham-handed way to proceed is to prefix "it is not the case that." We'll want to translate "$\Box$" by prefixing "It is necessary that," but to really specify what the modal formula is to mean, we'll need to specify, explicitly or tacitly, what flavor of necessity we are talking about. For now, that's logical necessity.

If we're talking about strict logical necessity, to answer the question "Which sentences are necessary?" it doesn't matter what the atomic sentences means. No matter how we interpret the atomic sentences, "$\Box\ \varphi$" is going to be true if φ is a tautology and false if it isn't. To my way of thinking, that's only half right. If φ is a tautology, $\Box\ \varphi$ should be true, but if we take "P" to mean, "If Cherry is brown and furry, Cherry is furry and brown," or to mean "Either there are no Greek philosophers or some philosophers are Greek," "$\Box\ \varphi$" ought to count as true.

Let us agree to call sentences like "A triangle has three sides" and "A gander is a male goose" logically necessary. They are made true by the rules of the language, though they are not made true by the rules that govern the logical connectives. A definition is a rule that specifies how the defined term is to be used.Our completeness theorem for Strict S5 consisted in devising a formal language, writing down some rules for the language, then inquiring what results from the rules we just wrote down.

I'll be taking it for granted that standard sentential calculus accurately describes the semantics of "or," "and," and "not." We'll focus all our attention on the modal operators. Paul Grice[2] gives strong reasons to think that the logic-book symbolizations of "$\vee$," "$\wedge$," and "~" are accurate. As we'll see, his defense of the accuracy of "$\rightarrow$" is harder to sustain.

We would like to identify the logic of logical necessity. We seek to identify the modal formulas with the property that every English sentence obtained by interpreting the sentence is true.

The reasoning we went through before gives us reason to accept that the formal-language sentences that are true under every interpretation include all the members of S5. The question we want to address is whether there are further formulas of the MSC, beyond those in S5, that are also true under every interpretation? We'll argue that there are not.

Before we undertake the proof, let's look at an example within the language whose only atomic sentences are "P" and "Q." The axioms of S5 allow models in which "(~ P $\wedge$ ~ Q)" is treated as impossible, whereas "(P $\wedge$ Q)," "(P $\wedge$ ~ Q)," and "(~ P $\wedge$ ~ Q)" are all treated as possible. Such models are allowed by the modal formalism, without taking account of how the formalism is interpreted. Can we find a way to interpret the formal language into English that preserves the modal status of these sentences, while treating necessity as logical necessity? To do so, what is required is to find appropriate translations of "P" and "Q."

Here's a way to do it: Translate "P" as "There is a dog with exactly one flea," and translate "Q" as "Either there is a dog with exactly two fleas or there is no dog with exactly one flea." This translation takes the first three sentences to English sentences that are logically equivalent to "There is a dog with exactly one flea and a dog with exactly two fleas," "There is a dog with exactly one flea and no dog with exactly two fleas," and "There is no dog with exactly one flea," respectively; these are all logically contingent. The English sentence that translates "(~ P $\wedge$ ~ Q)" is logically inconsistent.

Let us turn to the general proof. We already know that, if we translate a modal formula that's in S5 into English, translating "$\vee$," "$\wedge$," and "~" in the usual way, and translating "$\Box$" as "it is logically necessary that," the result will be a true English sentence. We want to see the converse: If χ is a modal formula that isn't in S5, it will be possible to translate it into a false English sentence. Let's suppose that atomic components of χ are among $\alpha_1,\ldots, \alpha_n$, Since χ isn't in S5, there is a simple Kripke model <W,I,@> for the language with those n atomic sentences in which χ is false.

The first part of the proof just involves modal sentential calculus, without bringing English translations into the picture. We want to show that the following holds, for each $k \leq n$:

[2] "The Logic of Conversation," *Studies in the Way of Words* (Harvard University Press: 1989), pp. 1-143.

There is a substitution s, defined on the little language whose atomic sentences are $\alpha_1,\ldots,\alpha_k$, such that, whenever σ is a state description for the little language, the following holds:

If σ is necessary in the model <W,I,@>, $s(\sigma)$ is tautological
If σ is impossible in the model, $s(\sigma)$ is truth-functionally inconsistent.
If σ is contingent in the model, $(s(\sigma) \leftrightarrow \sigma)$ is necessary.

We prove is by induction on k. For k = 1, we define:

$s(\alpha_1) = \top$ if α_1 is necessary in the model
$= \bot$ if α_1 is impossible in the model
$= \alpha_1$ if α_1 is contingent in the model

Given $s(\alpha_1),\ldots, s(\alpha_k)$, define $s(\alpha_{k+1})$ to be the following:

$((\alpha_{k+1} \vee \beta) \wedge \gamma)$,

where β is the disjunction of all the sentences $s(\sigma)$ such that σ a state description for $\alpha_1,\ldots, \alpha_k$ with $(\sigma \wedge \sim \alpha_{k+1})$ impossible in the model, and γ is the conjunction of all the $\sim s(\sigma)$s such that σ is a state description for $\alpha_1,\ldots, \alpha_k$ with $(\sigma \wedge \alpha_{k+1})$ impossible in the model. The proof that, for ρ a state description for $\alpha_1,\ldots, \alpha_k$, $(\rho \wedge \alpha_{k+1})$ and $(\rho \wedge \sim \alpha_{k+1})$ have the desired properties is a slog, going through case after tedious case. I won't go through it here.

This is the formal-language part of the proof. To apply it to natural languge, we need talk about translation from the formal language into English. We's set $t(\varphi \vee \psi)$ equal to the result of inserting "or" between $t(\varphi)$ and $t(\psi)$. Similarly for "$\wedge$" and "and." We'll translate "~" crudely, by prefixing "it is not the case that," and we'll translate "$\Box$" by prefixing "it is logically necessary that."

It remains to deal with the atomic sentences. For that we need a supply of n contingent sentences that are independent, so that in figuring out whether one of them is true, it's no help to know which of the others are true. For that purpose, we'll use the sentences "There is a dog with exactly i fleas." If α_i is true in @, we'll take $t(\alpha_i)$ to be whichever of "There is a dog with exactly i fleas" and "There no dog with exactly i fleas" is true. If α_i is false in @, we'll take $t(\alpha_i)$ to be whichever of "There is a dog with exactly i fleas" and "There is no dog with exactly i fleas" is false.

If the state description σ is necessary in <W,I,@>, $t(s(\sigma))$ is tautological. If σ is contingently true in the model, $t(s(\sigma))$ will be contingently true. If σ is contingently false in the model, $t(s(\sigma))$ will be contingently false. If σ is impossible in the model, $t(s(\sigma))$ will be truth-functionally inconsistent.

It follows that the model description of <W,I,@> is true. The S5 proof that the model description of <W,R,I> implies ~ χ carries over under the translation $t \circ s$, defined by $t \circ s(\varphi) = t(s(\varphi))$. It follows that $t \circ s(\chi)$ is false.

The translation t respects truth in the model – it takes modal formulas that are true in the model to true English sentences and sentences that are false in the model to false English sentences – but it doesn't respect the model's judgments about necessity and possibility. It takes theorems of S5 to necessarily true English sentences and modal formulas that are refutable in S5 to necessarily false English sentences, but modal formulas not in S5 that the model regards as necessary will be taken by t to contingently true English sentences, and modal formulas not refutable in S5 that the model regards as impossible will be taken by t to English sentences that are contingently false.

The adjusted translation scheme $t \circ s$ respects the model's judgments about necessity and possibility as well as its judgments about truth. It takes modal formulas that the model regards as necessary to logically necessary English sentences, and it take modal formulas that the model regards as impossible to logically impossible English sentences. It takes modal formulas that the model regards as contingently true to contingently true English sentences, and it takes modal formulas that the model regards as contingently false to contingently false English sentences.

We see just by examining the modal axioms and rules that, if a modal formula χ is in S5, any English translation that gives the standard treatment to "∨," "∧," and "~" and translates "□" as logical necessity will take χ to a true sentence of English. We now see that, if χ isn't in S5, there is a translation that takes χ to a false English sentence. So there is an exact matchup between derivability in S5 and truth under every translation.⊠