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SEMANTICAL ANALYSIS OF MODAL LOGIC I
NORMAL MODAL PROPOSITIONAL CALCULI

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The present paper attempts to extend the results of [1], in the domain of the propositional calculus, to a class of modal systems called "normal." This class includes M and S4 as well as S5; we will also treat a new system, the "BROUWERsche" system. In sequels to the present paper, we intend to extend the treatment to non-normal modal propositional calculi (see [7] for an enumeration of the systems included), and to quantificational and identity extensions of all these propositional systems of modal logic (see again [7] for details; but readers of [1] will have an inkling of how quantificational extensions are to be carried out). Thorough acquaintance with [1] is presupposed; and many of the proofs in this paper (which, by reason of the many systems here treated, is occasionally somewhat compressed), are better comprehended by a comparison with the corresponding proofs in [1].

1. Normal modal propositional calculi

A *modal propositional calculus* (MPC) is given by a denumerably infinite list of *propositional variables* $P, Q, R, \ldots$, which can be combined, using the *connectives* $\wedge$, $\sim$, $\Box$, to form *formulae* (wffs) as in [1]. (The propositional variables are thus the *atomic formulae* of the systems. Below we will use the letters $P, Q, R, \ldots$, as *metavariables* ranging over atomic formulae; $A, B, C, \ldots$, as metavariables over arbitrary formulae.) A modal propositional calculus is called *normal* iff it contains as theorems the axiom schemes A1 and A3 of [1], and contains as admissible (derivable) rules the two rules of inference R1 and R2 of [1]:

A1. $\Box A \supset A$

A3. $\Box(A \supset B) .\supset. \Box A \supset \Box B$

R1. If $\vdash A$ and $\vdash A \supset B$, $\vdash$ then B.

R2. If $\vdash A$, then $\vdash \Box A$.

(The non-normal systems to be considered in another paper will fail to satisfy R2; in the paper on quantificational extensions we will also consider systems that are non-normal in the sense that they are modified in the direction of PRIOR's Q.)

The system M (T) of FEYS- VON WRIGHT (cf. [3], [19]), is given by the axioms A1 and A3, and the rules R1 and R2. The system S4 is obtained by adding to M

A4. $\vdash \Box A \supset \Box \Box A$

as an axiom scheme. The BROUWERsche *axiom* (cf. [4], p. 497) is the scheme:

$$A \supset \Box \Diamond A.$$

The BROUWERsche *system* is obtained by adding the BROUWERsche axiom to M. Finally, S5 is defined as in [1]; i.e., it is M plus the scheme:

A2. $\sim \Box A \supset \Box \sim \Box A.$

It is known (see the appendix to [4]) that S4 plus the BROUWERsche axiom is equivalent to S5. The present paper will make it clear that this theorem is essentially equivalent to one which is better known and simpler: A reflexive, transitive, and symmetric relation partitions its field into disjoint equivalence classes (cf. 2.1, next to last remark; also 2.2).

2. Normal models

A *normal model structure* (n. m. s.) is an ordered triple $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$, where K is a non-empty set, $\mathsf{G} \in \mathsf{K}$, and $\boldsymbol{R}$ is a reflexive relation defined on K. If $\boldsymbol{R}$ is transitive, we call the n.m.s. an S4 *model structure*; if $\mathfrak{R}$ is symmetric, we call it a BROUWERsche *model structure*; if $\boldsymbol{R}$ is an equivalence relation, we call it an S5 *model structure*. A normal model structure is also called an M *model structure*. In this paper the adjective "normal" will often be omitted, and we will speak simply of a "model structure" (m.s.).

An M (S4, S5, BROUWERsche) *model* for a wff A of M (S4, S5, the BROUWERsche system) is a binary function $\Phi(P, \mathsf{H})$ associated with a given M (S4, S5, BROUWERsche) model structure $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$. The first variable '$P$' ranges over atomic subformulae of A, while the second variable 'H' ranges over the members of K. The range of Φ is the set $\{\mathsf{T}, \mathsf{F}\}$; i.e., $\Phi(P, \mathsf{H}) = \mathsf{T}$ or $\Phi(P, \mathsf{H}) = \mathsf{F}$.

Now given a model Φ associated with a m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$, we will define for any subformula B of A, and any $\mathsf{H} \in \mathsf{K}$, a value $\Phi(B, \mathsf{H})$ (which will be T or F); *i.e.*, we define a unique extension of Φ in which the first argument ranges over *all* subformulae of A, not merely atomic subformulae. If B is atomic, (i.e., is a propositional variable), the corresponding value $\Phi(B, \mathsf{H})$ has already been defined. For more complex formulae we define the valuation by induction on the number of connectives in the formula. Assume that $\Phi(B, \mathsf{H})$ and $\Phi(C, \mathsf{H})$ have already been defined for each $\mathsf{H} \in \mathsf{K}$. If $\Phi(B, \mathsf{H}) = \Phi(C, \mathsf{H}) = \mathsf{T}$, then $\Phi(B \wedge C, \mathsf{H}) = \mathsf{T}$; otherwise $\Phi(B \wedge C, \mathsf{H}) = \mathsf{F}$. If $\Phi(B, \mathsf{H}) = \mathsf{T}$, then $\Phi(\sim B, \mathsf{H}) = \mathsf{F}$; otherwise if $\Phi(B, \mathsf{H}) = \mathsf{F}$, $\Phi(\sim B, \mathsf{H}) = \mathsf{T}$. Finally, to define $\Phi(\Box B, \mathsf{H})$: If $\Phi(B, \mathsf{H}') = \mathsf{T}$ for every H' in K such that $\mathsf{H}\,\boldsymbol{R}\,\mathsf{H}'$, we say $\Phi(\Box B, \mathsf{H}) = \mathsf{T}$; otherwise, if there exists H' such that $\mathsf{H}\,\boldsymbol{R}\,\mathsf{H}'$ and $\Phi(B, \mathsf{H}') = \mathsf{F}$, we say $\Phi(\Box B, \mathsf{H}) = \mathsf{F}$.

We say a formula A is *true* in a model Φ associated with a m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ if $\Phi(A, \mathsf{G}) = \mathsf{T}$; *false*, if $\Phi(A, \mathsf{G}) = \mathsf{F}$. We say A is *valid* if it is true in all its models[1]); *satisfiable* if it is true in at least one of them. We say shall show below (completeness and consistency theorems) that a formula is valid if and only if it is provable in the appropriate system.[2])

2.1. Informal explanation

In [1] the writer introduced a modelling for S5 based on the notion of a "possible world". We were given a set K of possible worlds, with one element G singled out as the "real" world. A proposition was to be necessary iff it was "true in all possible worlds".

The present treatment generalizes that of [1] in the following respects: (1) Again we have a set K of "possible worlds"; again the real world G is a distinguished element. Every atomic formula (i.e., propositional variable) P is assigned a truth-value in each world H; in fact, this truth-value is $\Phi(P, \mathsf{H})$. Here we already have a slight divergence from the treatment in [1]. For in [1], we did not have an auxiliary function Φ to assign a truth-value to P in the world H; instead H itself was a "complete assignment", that is, a *function* assigning a truth-value to every atomic subformula of a formula A. On this definition, "worlds" and complete assignments are identified; so distinct worlds give distinct complete assignments. This last clause means that *there can be no two worlds in which the same truth-value is assigned to each atomic formula.* Now this assumption turns out to be convenient perhaps for S5, but it is rather inconvenient when we treat normal MPC's in general. In the present paper we drop it; we are given an arbitrary set K of "possible worlds",

1) Actually, we define validity in M (S4, S5, BROUWERsche) as truth in all M (S4, S5, BROUWERsche) models. Explicit mention of a particular system, M, S4, S5, or BROUWERsche, is omitted here and henceforth whenever the same remarks or definitions apply to all four systems. It will be understood that for "model" or "m.s." we read M, S4, S5, or BROUWERsche model or m.s., the other definitions being correspondingly relativized to a particular system.

2) For systems based on S4 and M and (with his initial formulation modified; see below) on S5, HINTIKKA has discovered a modelling similar to the present one. T. J. SMILEY and his pupils have discovered modelling for these three systems, based on MCKINSEY [9], which, though somewhat further removed, is probably basically equivalent to the one given here; and I have heard lately that MCKINSEY himself left an unpublished modelling of his own. BAYART [8] has proved the completeness of S5* independently of [1]. GUILLAUME [8] has used semantic tableaux in a topological investigation of M and S4; GENTZEN rules similarly to the tableau rules are given in [10], [11], [12]. The modelling for modal logic given in KANGER [12], though more complex, is similar to that in the present paper. The most surprising anticipation of the present theory, discovered just as this paper was almost completed, is the algebraic analogue in JONSSON and TARSKI [17]. Independently and in ignorance of [17] (though of course much later), the present writer derived its main theorem by an algebraic analogue of his semantical methods; the proof will appear elsewhere. None of these authors (except for some initial impetus from CURRY [10]) has been compared in detail by the present writer with his own work, which is independent of them; a detailed comparison may be useful to others.

a distinguished "real world" G, and a function $\Phi(P, \mathsf{H})$ assigning to each proposition P a truth-value in the world H. (2) A deviation from [1] of more consequence is found in the use of the relation $\boldsymbol{R}$. Intuitively we interpret the relation $\boldsymbol{R}$ as follows: Given any two worlds $\mathsf{H}_1, \mathsf{H}_2 \in \mathsf{K}$, we read "$\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_2$" as H_2 is "possible relative to H_1", "possible in H_1", or "related to H_1"; that is to say, every proposition true in H_2 is to be possible in H_1. Thus the "absolute" notion of possible world in [1] (where every world was possible relative to every other) gives way to relative notion, of one world being possible relative to another. It is clear that every world H is possible relative to itself; for this simply says that every proposition *true* in H is also *possible* in H. In accordance with this modified view of "possible worlds", we evaluate a formula A as *necessary* in a world H_1 if it is *true* in every world possible relative to H_1; i.e., $\Phi(\Box A, \mathsf{H}_1) = \mathsf{T}$ iff $\Phi(A, \mathsf{H}_2) = \mathsf{T}$ for each H_2 such that $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_2$. Dually, A is possible in H_1 iff there exists H_2, possible relative to H_1, in which A is *true*.

Finally, we can ask various questions regarding the relation $\boldsymbol{R}$, e.g., whether it is transitive. Given $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_2$ and $\mathsf{H}_2 \boldsymbol{R} \mathsf{H}_3$, does it follow that $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_3$? To say that $\mathsf{H}_2 \boldsymbol{R} \mathsf{H}_3$ is to say that any formula A *true* in H_3 is possible in H_2 (i.e., $\Diamond A$ is true in H_2); but then, since $\mathsf{H}_1 \mathfrak{R} \mathsf{H}_2$, it follows in turn that $\Diamond A$ is possible (A is "possibly possible" and $\Diamond\Diamond A$ is true) in H_1. In order to assert that $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_3$, we need to show that if A is true in H_3, it is possible in H_1; but we have shown above that A is at least *possibly* possible in H_1; so the additional reduction axiom we need in order to assert $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_3$ is "what is possibly possible is possible"! This reduction axiom of S4 boils down to the assertion that $\boldsymbol{R}$ is transitive. Similarly, the BROUWERsche axiom says that $\boldsymbol{R}$ is symmetric. For let $A \supset \Box\Diamond A$ hold and let $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_2$; then we will have $\mathsf{H}_2 \boldsymbol{R} \mathsf{H}_1$ if we can show that anything *true* in H_1 is possible in H_2. But if A is true in H_1, by the BROUWERsche axiom $\Diamond A$ is necessary in H_1; that is, it is true in all worlds possible relative to H_1. In particular, $\Diamond P$ is true in H_2, Q.E.D. *The reduction axioms of classical modal logic reduce to simple properties (above and beyond reflexivity) of the relation* $\boldsymbol{R}$. If we abandon the relation $\boldsymbol{R}$ and just use the set K as in [1] (or equivalently, we let $\boldsymbol{R}$ be the relation holding between every pair of elements of K), then we are saying that every possible proposition is necessarily possible, the characteristic axiom of S5. It turns out that we get the same reduction axiom, however, if we simply assume that $\boldsymbol{R}$ is an equivalence relation; see 2.2 below.

(3) One minor deviation from [1]: In the present paper, if $\Phi(B, \mathsf{G}) = \mathsf{T}$, we say that B is *true* in the model Φ; previously we said B was *valid* in the model. The present terminology is clearly an improvement.

2.2. Connected models

Let $\boldsymbol{R}^*$ be the "ancestral" of $\boldsymbol{R}$, in the sense of [2].[1]) A m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ is called *connected* iff for all $\mathsf{H} \in \mathsf{K}$, $\mathsf{G}\, \boldsymbol{R}^* \mathsf{H}$. A model Φ is *connected* if it is defined on a

[1]) Similarly for the relation "S" below, S^* will be its ancestral.

connected model structure. We show that *every satisfiable formula has a connected model* (equivalently, that every non-valid formula has a connected countermodel). (Here if A is a formula, Φ is a model for A iff A is true in Φ; otherwise, a countermodel.)

Let A be satisfiable in a model $\Phi(P, \mathsf{H})$ defined on a m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$. Let K' be the set of all $\mathsf{H} \in \mathsf{K}$ such that $\mathsf{G}\ \boldsymbol{R}^*\ \mathsf{H}$, let $\boldsymbol{R}'$ be the restriction of $\boldsymbol{R}$ to K', and let $\Phi'(P, \mathsf{H})$ be Φ with H restricted to K'. Then $(\mathsf{G}, \mathsf{K}', \boldsymbol{R}')$ is a m.s., and Φ' is a model in $(\mathsf{G}, \mathsf{K}', \boldsymbol{R}')$. Clearly Φ' is connected. We show by induction that for any subformula B of A, and $\mathsf{H} \in \mathsf{K}'$, $\Phi'(B, \mathsf{H}) = \Phi(B, \mathsf{H})$. (Hence it will follow that, since $\Phi(A, \mathsf{G}) = \mathsf{T}$, $\Phi'(A, \mathsf{G}) = \mathsf{T}$, so that Φ' is a model of A as desired.) It B is atomic, the result is immediate. If the result has already been proved for C and D, and B is $C \wedge D$ or $\sim C$, the verification for B is trivial. If B is $\Box C$, we carry out the induction step thus: We notice that, if $\mathsf{H} \in \mathsf{K}'$, $\mathsf{H}\ \boldsymbol{R}'\ \mathsf{H}'$ implies $\mathsf{H}' \in \mathsf{K}'$, and hence $\mathsf{H}\ \boldsymbol{R}\ \mathsf{H}'$. So, for $\mathsf{H} \in \mathsf{K}'$, $\mathsf{H}\ \boldsymbol{R}\ \mathsf{H}'$ iff $\mathsf{H}\ \boldsymbol{R}'\ \mathsf{H}'$. By the inductive hypothesis, for $\mathsf{H}' \in \mathsf{K}'$, $\Phi(C, \mathsf{H}') = \Phi'(C, \mathsf{H}')$. Now (1) $\Phi(\Box C, \mathsf{H}) = \mathsf{T}$ iff $\forall \mathsf{H}' \in \mathsf{K}$ s.t. $\mathsf{H}\ \boldsymbol{R}\ \mathsf{H}', \Phi(C, \mathsf{H}') = \mathsf{T}$; (2) $\Phi'(\Box C, \mathsf{H}) = \mathsf{T}$ iff $\forall \mathsf{H}' \in \mathsf{K}'$ s.t. $\mathsf{H}\ \boldsymbol{R}\ \mathsf{H}', \Phi'(C, \mathsf{H}') = \mathsf{T}$. – The preceding discussion shows that if $\mathsf{H} \in \mathsf{K}'$, he right hand sides of (1) and (2) are equivalent; so $\Phi(\Box C, \mathsf{H}) = \mathsf{T}$ iff $\Phi'(\Box C, \mathsf{H}) = \mathsf{T}$, and hence $\Phi(\Box C, \mathsf{H}) = \Phi'(\Box C, \mathsf{H})$, as desired.

So without loss of generality, we could restrict our considerations to *connected* models. Note that in a connected model in which $\boldsymbol{R}$ is an equivalence relation, *any* two worlds are related. This fact accounts for the adequacy, for S5, of the model theory of [1].

2.3. Trees

A triple $(\mathsf{G}, \mathsf{K}, \boldsymbol{S})$, with K a set, $\mathsf{G} \in \mathsf{K}$, and $\boldsymbol{S}$ a relation defined on K (not necessarily reflexive) is called a *tree* (and G is called its *origin* iff: (1) There is no $\mathsf{H} \in \mathsf{K}$ s.t. $\mathsf{H}\ \boldsymbol{S}\ \mathsf{G}$; (2) for every $\mathsf{H} \in \mathsf{K}$ except G, there is a unique H' s.t. $\mathsf{H}'\ \boldsymbol{S}\ \mathsf{H}$; (3) for every $\mathsf{H} \in \mathsf{K}$, $\mathsf{G}\ \boldsymbol{S}^*\ \mathsf{H}$. If $\mathsf{H}\ \boldsymbol{S}\ \mathsf{H}'$, we call H the *predecessor* of H'; in terms, then of $\boldsymbol{S}$, K is characterized as the field of $\boldsymbol{S}$, and G as the unique element of K without a predecessor. So we can speak of a relation $\boldsymbol{S}$ as a *tree* relation if a G and K satisfying the previous conditions exist; they will then be determined by $\boldsymbol{S}$.

An M-m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ is called a *tree* M-m.s. iff there exists a relation $\boldsymbol{S}$ such that $(\mathsf{G}, \mathsf{K}, \boldsymbol{S})$ is a tree and $\boldsymbol{R}$ is the smallest reflexive relation containing $\boldsymbol{S}$ (the reflexive relation "generated by" $\boldsymbol{S}$). Clearly in this case $\mathsf{H}_1\ \boldsymbol{R}\ \mathsf{H}_2$ iff $\mathsf{H}_1\ \boldsymbol{S}\ \mathsf{H}_2$ or $\mathsf{H}_1 = \mathsf{H}_2$. Similarly an S4 (Brouwersche, S5) m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ is a *tree* S4 (Brouwersche, S5) m.s. iff there is a relation $\boldsymbol{S}$ such that $(\mathsf{G}, \mathsf{K}, \boldsymbol{S})$ is a tree and $\boldsymbol{R}$ is the smallest reflexive and transitive (reflexive and symmetric, equivalence) relation containing $\boldsymbol{S}$. Note that an S4 m.s., may be, a tree S4-m.s., and yet not be a tree M-m.s.; and similarly for the other cases.

Clearly every tree m.s. is connected. For by condition 3), for each $\mathsf{H} \in \mathsf{K}$, $\mathsf{G}\ \boldsymbol{S}^*\ \mathsf{H}$; and since $\boldsymbol{S} \subseteqq \boldsymbol{R}$, it follows that $\boldsymbol{S}^* \subseteqq \boldsymbol{R}^*$. In S5, every finite or countable connected m.s. is a tree S5-m.s. This need not hold for S4, and in fact there are connected S5 model structures (e.g., $\mathsf{K} = \{\mathsf{G}, \mathsf{H}\}$ and $\boldsymbol{R}$ relates all pairs) which are tree S5-model structures, but not tree S4-model structures. Nevertheless, when confusion does not arise, we will leave out reference to a given system when that system is understood throughout; if we say "tree m.s." when we are talking about S4, we mean "tree S4-m.s.," and the like.

A model associated with a tree m.s. is called a tree model. We will show below (a stronger result than 2.2) that the semantical theory would lose no generality if only tree models were admitted (cf. 3.3). Tree model structures admit the obvious convenient diagramatic representation which inspires their name. Put G at the origin, connect each H such that $\mathsf{G}\ S\ \mathsf{H}$ directly to G, and so on.

3. Semantic tableaux

The notion of semantic tableau (cf. BETH [5]) developed here is similar to that of [1], which should be read as background. Again we deal at each stage of the construction with a *system* of *alternative sets* of tableaux; in each set, one tableau is singled out as the main tableau, while the others are *auxiliary*. The only difference between the present situation and that of [1] lies in the fact that each alternative set of the system is ordered by a reflexive relation R, parallel to the reflexive $\boldsymbol{R}$ of the model theory, so that each stage of the construction is now a system of *ordered* alternative sets. We use letters $\mathfrak{t}, \mathfrak{t}', \mathfrak{t}'', \mathfrak{t}_1, \mathfrak{t}_2, \ldots$ for tableaux; if $\mathfrak{t}_1\ R\ \mathfrak{t}_2$, we say that $\mathfrak{t}_2$ is "related to" $\mathfrak{t}_1$, or that $\mathfrak{t}_2$ is "auxiliary to" $\mathfrak{t}_1$. The rules Nl, Nr, and Λl remain as in [1], as we shall see. So, in effect, does the rule Λr, but its restatement is complicated (see below). The rules Yl and Yr are changed so as to parallel the new treatment of necessity in the model theory.

Given a formula A, in order to see whether it is valid we attempt to find a *countermodel* to that formula; if no countermodel exists, the formula is valid. If A has the form $A_1 \wedge \cdots \wedge A_m .\supset. B_1 \vee \cdots \vee B_n$, clearly $A_1, \ldots, A_m$ must be *true*, and $B_1, \ldots, B_n$ *false*, in any countermodel to A. We represent this situation by putting $A_1, \ldots, A_m$ on the left, and $B_1, \ldots, B_n$ on the *right* of the main tableau, of the construction; this represents our attempt to find a model in which $A_1, \ldots, A_m$ are true while $B_1, \ldots, B_n$ are false. We then continue the construction by the following rules (which apply to any tableau, main or auxiliary):

Nl. If $\sim A$ appears in the left column of a tableau, put A in the right column of that tableau.

Nr. If $\sim A$ appears in the right column of a tableau, put A in the left column of that tableau.

Λl. If $A \wedge B$ appears in the left column of a tableau, put A and B in the left column of that tableau.

Λr. If $A \wedge B$ appears in the right column of a tableau $\mathfrak{t}$, there are two alternatives; Extend the tableau $\mathfrak{t}$ either by putting A in the right column or by putting B in the right column. If the tableau t is in an ordered set $\mathscr{S}$, it is clear that the next stage we have two *alternative* sets, depending on which extension of the tableau t is adopted. Informally speaking, if the original ordered set is diagrammed structurally on a sheet of paper, we copy over the entire diagram twice, in one case putting in addition A in the right column of the tableau t and in the other case putting B; the two new sheets correspond to the two new alternative sets. I hope this explanation makes the process clear intuitively; the formal statement is rather messy: Given a tableau t in an alternative set $\mathscr{S}$, if t has $A \wedge B$ on the right, we replace $\mathscr{S}$ by two alternative sets, $\mathscr{S}_1$ and $\mathscr{S}_2$, where $\mathscr{S}_1 = \mathscr{S} - \{t\} \cup \{t_1\}$ and $\mathscr{S}_2 = \mathscr{S} - \{t\} \cup \{t_2\}$, and $t_1 (t_2)$ is like t except that in addition it contains $A(B)$ on the right. Since $\mathscr{S}$ is ordered by a reflexive relation R, we must define orderings R_1 and R_2 on the two new sets, $\mathscr{S}_1$ and $\mathscr{S}_2$. Informally stated, the ordering $R_1 (R_2)$ of $\mathscr{S}_1 (\mathscr{S}_2)$ is precisely the same as that of $\mathscr{S}$, except that $t_1 (t_2)$ replaces t throughout. We state this condition more formally for $\mathscr{S}_1$: Let t′ or t″ be any tableau of $\mathscr{S}$ other than t. Then $t' R_1 t_1$ iff $t' R t$ (in $\mathscr{S}$), $t_1 R_1 t'$ iff $t_1 R t$, and $t' R_1 t''$ iff $t' R t''$. *Further, to make R_1 reflexive, we stipulate that* $t_1 R t_1$. These conditions determine the new ordering R_1 on $\mathscr{S}_1$. Similarly for $\mathscr{S}_2$.

Yl. If $\Box A$ appears on the left of a tableau t, then for every tableau t′ such that $t R t'$, put A on the left of t′.

Yr. If $\Box A$ appears on the right of a tableau t, then we start out a new tableau t′, with A on the right, and such that $t R t'$.

Given any alternative set $\mathscr{S}$, ordered by a relation R, the rules above stipulate that certain tableaux are to be R-related (cf. in particular Yr and Λr). In addition to these stipulations, we set requirements corresponding to those for the corresponding model structures. As $\boldsymbol{R}$ in $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ was reflexive, so R is assumed reflexive. Further for S4-tableaux we assume R to be transitive, in BROUWERsche tableaux we assume R to be symmetric, and in S5-tableaux we assume both. In M-tableaux, of course, we place no restriction, other than reflexivity, on R. Finally we assume that R holds only as required by the stipulations preceding and by the rules Yr and Λr above (i.e., R is to be the smallest relation satisfying these conditions).

As in [1], we define a tableau as *closed* iff some formula A appears on both sides of the tableau, a set of tableaux as closed iff some tableau in it is closed, a system of tableaux as closed iff each of its alternative sets is closed. Since at each stage of the construction we have a system of alternative sets, we can finally define a construction to be clos d iff at some stage of the construction, a closed system of alternative sets appears.

Finally, we define (terminology of GALLEGHER) a construction *for A* as one started out by putting A on the right of the main tableau of the construction.

Two restrictions are placed on the rules, in order to facilitate termination of the construction. A rule is not to be applied to a formula occurring in a closed alternative set; nor is it to be applied if it is "superfluous". ("Superfluous" is defined by example: Yr is superfluous iff there already exists a tableau t′ s.t. t R t′ with A os the right of t′; this tableau t′ may, of course, be t itself. Nl is superfluous iff A already appears on the right of t, and so on.)

Strictly speaking, it might be more rigorous if we specified a definite order of priority in which the rules were to be applied. But actually (as is clear from the semantical results of 3.2), such a restriction of order would be irrelevant to the question whether a tableau construction closes; the rules are "permutable." Hence on the other hand, if it is convenient for a particular proof, we can specify any ordering we desire; this fact is exploited in 5.1.

3.1. Examples

The following is an S4-construction beginning with $\Box(A \wedge B)$ on the left and $\Box(\Box A \wedge \Box B)$ on the right:

$$
\begin{array}{r|l} \Box(A \wedge B) & \Box(\Box A \wedge \Box B) \\ A \wedge B & \\ A & \\ B & \\ \multicolumn{2}{c}{\qquad\quad t_1} \end{array}
\quad \rightarrow \quad
\begin{array}{l|l} \qquad & \Box A \wedge \Box B \\ & \\ & \\ & \\ \multicolumn{2}{c}{t_2} \end{array}
$$

The first formula on the left of t_1 is given; the second is obtained by Yl (remember that R is reflexive!), and the third and fourth by Λl. Applying Yr on the right, we start out the tableau t_2 as shown. The arrow indicates that $t_1 \, R \, t_2$.

At this point the rule Λr admits of two alternatives. Put these alternatives down as follows:

$$
\begin{array}{r|l} \Box(A \wedge B) & \Box(\Box A \wedge \Box B) \\ A \wedge B & \\ A & \\ B & \\ \multicolumn{2}{c}{\qquad\quad t_1} \end{array}
\quad \rightarrow \quad
\begin{array}{l|l} \qquad & \Box A \wedge \Box B \\ & \Box A \\ & \\ & \\ \multicolumn{2}{c}{t_{21}} \end{array}
$$

or:

$$
\begin{array}{r|l} \Box(A \wedge B) & \Box(\Box A \wedge \Box B) \\ A \wedge B & \\ A & \\ B & \\ \multicolumn{2}{c}{\qquad\quad t_1} \end{array}
\quad \rightarrow \quad
\begin{array}{l|l} \qquad & \Box A \wedge \Box B \\ & \Box B \\ & \\ & \\ \multicolumn{2}{c}{t_{22}} \end{array}
$$

On each alternative, we recopy the entire diagram, but in one, $\Box A$ goes on the right of t_2 (which gets relabelled t_{21}), while in the other $\Box B$ goes on the right of t_2 (which gets relabelled t_{22}). We continue the development of the first alternative (the other is similar):

$$
\begin{array}{c|l}
\Box(A \wedge B) & \Box(\Box A \wedge \Box B) \\
A \wedge B & \\
A & \\
B & \\
\multicolumn{2}{c}{t_1}
\end{array}
\quad \rightarrow \quad
\begin{array}{c|l}
A \wedge B & \Box A \wedge \Box B \\
A & \Box A \\
B & \\
\multicolumn{2}{c}{t_{21}}
\end{array}
\quad \rightarrow \quad
\begin{array}{c|l}
(A \wedge B) & A \\
(A) & \\
(B) & \\
\multicolumn{2}{c}{t_3}
\end{array}
$$

By Yr again, we introduce t_3 with A on the right. We have $t_1 R t_2$, $t_{21} R t_3$ as shown by the arrow. Since $t_1 R t_{21}$ and $\Box(A \wedge B)$ is on the left of t_1, we put $A \wedge B$ on the left of t_{21}. Also, by the transitivity of the arrow $t_1 R t_3$, so we put $(A \wedge B)$ on the left of t_3. But thence by Λl, A and B go on the left of t_3. The construction is closed since A appears on both sides of t_3. This shows that there can be no S4-model Φ in which $\Box(A \wedge B)$ is true while $\Box(\Box A \wedge \Box B)$ is false. For in such a model, as is clear by following the construction, to each t_i $(i = 1, 2, 3)$, there would correspond a world $H_i (G = H_1)$, with the property that for any C, $\Phi(C, H_i) = T(F)$ if C appears on the left (right) of t_i. Since on the alternative we have chosen, A appears on both left and right of t_3, we would have to have both $\Phi(A, H_3) = T$, and $\Phi(A, H_3) = F$, a contradiction. (On the other alternative, we would have $\Phi(B, H_3) = T = F$.) Note further that if R is not transitive, we would no longer have closure; in fact the parenthetical formulae would no longer appear in t_3. Thence we would indeed get an M-model Φ in which $\Phi(\Box A \wedge \Box B) = T$ while $\Phi(\Box(\Box A \wedge \Box B)) = F$. The nature of this model Φ in (H_1, K, R), with $H_1 R H_2$ and $H_2 R H_3$ can be "read off" (partially) from the tableaux. We look at those places where atomic formulae occur on the left or right. Since A and B are on the left of t_1 and t_{21}, we have $\Phi(A, H_1) = \Phi(B, H_1) = \Phi(A, H_2) = \Phi(B, H_2) = T$; while on the other hand, since A is on the right of t_3, $\Phi(A, H_3) = F$. B appears on neither side of t_3; this shows $\Phi(B, H_3)$ may be assigned arbitrarily. The reader can check that, no matter which value we give to $\Phi(B, H_3)$, we have

$$\Phi(\Box(A \wedge B), H_1) = T \quad \text{and} \quad \Phi(\Box(\Box A \wedge \Box B), H_1) = F.$$

Further, one notes that the tableau construction would not be altered if the arrow were read as symmetric; so the model would still work if we stipulated in addition that $H_2 R H_1$ and $H_3 R H_2$. Hence it follows we have a BROUWERsche model with the stated properties. The upshot of this discussion is: $\Box(A \wedge B) \supset \Box(\Box A \wedge \Box B)$ is valid in S4, but not in M and the BROUWERsche system.

As an exercise, let the reader consider the following S5-construction, beginning with $\sim \Box A$ on the right and $\Box \sim \Box A$ on the left:

$$
\begin{array}{r|l}
\sim \Box A & \Box \sim \Box A \\
A & \Box A \\
& \\
& \\
\multicolumn{2}{c}{t_1}
\end{array}
\quad
\begin{array}{c}
\nearrow \\
\\
\searrow
\end{array}
\quad
\begin{array}{c}
\begin{array}{r|l}
\Box A & \sim \Box A \\
A &
\end{array} \\
t_2 \\
\begin{array}{r|l}
A & A
\end{array} \\
t_3
\end{array}
$$

Since A appears on both the left and right of t_3, the construction is closed. Notice that we required symmetry of R to put A on the left of t_1, while symmetry and transitivity were required to put A on the left of t_3. Hence neither the BROUWERsche nor the S4-constructions would be closed, and in the S4-construction A would not appear on the left of t_1. This shows that in S5, but in none of the other systems we have considered, $\sim \Box A \supset \Box \sim \Box A$ is valid.

3.2. Equivalence of tableaux to models

This section shows that a construction for A is closed if and only if A is valid for each of the four systems we are considering (or indeed for any other systems in which precisely the same restrictions are put on R for both tableaux and models). The theorem reduces to two lemmas, similar to the first two lemmas of [1].

Lemma 1. *If the construction for A is closed, A is valid.*

Proof. Assume for *reductio ad absurdum* that A is not valid. Then there exists a model Φ in a model structure $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ such that $\Phi(A, \mathsf{G}) = \mathsf{F}$. Now we shall show, by induction on n, that for each n, at the nth stage of the construction, there is an alternative set $\mathscr{S}$ of the construction and a map α, mapping tableaux of $\mathscr{S}$ into elements of K, with the following property: *If* t *is a tableau of* $\mathscr{S}$, $\mathsf{H} = \alpha(\mathrm{t})$, *and B is any formula occurring on the left (right) of* t, *then* $\Phi(B, \mathsf{H}) = \mathsf{T}(\mathsf{F})$. *Furthermore, if* t_1 *and* t_2 *are in* $\mathscr{S}$, $\mathsf{H}_1 = \alpha(\mathrm{t}_1)$ *and* $\mathsf{H}_2 = \alpha(\mathrm{t}_2)$, *then* $\mathrm{t}_1 R\, \mathrm{t}_2$ *implies* $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_2$.

To carry out the induction, notice that it is obvious for $n = 1$. Here we have only one tableau t with A on the right; and if we set $\alpha(\mathrm{t}) = \mathsf{G}$, we have $\Phi(A, \mathsf{G}) = \mathsf{F}$, as required. Assume the result proved for the nth stage; then there is an alternative set $\mathscr{S}$ of the nth stage, and a map α, with the required properties.

Let us attempt now to extend the result to the $(n+1)$th stage. The $(n+1)$th stage must be obtained from the nth by one of the rules, which is applied to some tableau of some alternative set $\mathscr{S}'$ of the system at this stage. Now if $\mathscr{S}' \neq \mathscr{S}$, then $\mathscr{S}$ remains unchanged at the $(n+1)$th stage and the induction step has been verified trivially. So let us assume that $\mathscr{S}' = \mathscr{S}$, so that the rule is applied to some tableau t of $\mathscr{S}$. It the rule is Λl, then $B \wedge C$ appears on the left of t, and by hypothesis, for $\mathsf{H} = \alpha(\mathrm{t})$, we have $\Phi(B \wedge C, \mathsf{H}) = \mathsf{T}$. Hence $\Phi(B, \mathsf{H}) = \Phi(C, \mathsf{H}) = \mathsf{T}$, so when Λl instructs us to put both B and C on the left column of t, it preserves the required properties of α. We say that this fact "validates" Λl. Similarly we can validate Nl and Nr. If the rule applied is Λr, then $B \wedge C$ appears on the right of t, so by hypothesis $\Phi(B \wedge C, \mathsf{H}) = \mathsf{F}$. Hence either $\Phi(B, \mathsf{H}) = \mathsf{F}$ or $\Phi(C, \mathsf{H}) = \mathsf{F}$. Now Λr correctly instructs us to consider these two possibilities; it has us replace the tableau t of $\mathscr{S}$ by either of two alternative tableaux, t_1 and t_2, both like t except that in addition t_1 contains A and t_2 contains B on the right, yielding two new alternative sets $\mathscr{S}_1$ and $\mathscr{S}_2$. If $\Phi(B, \mathsf{H}) = \mathsf{F}$, the set $\mathscr{S}_1$ will satisfy all requirements; otherwise, $\Phi(C, \mathsf{H}) = \mathsf{F}$, and $\mathscr{S}_2$ satisfies all requirements. If the rule Yr is applied to a tableau t with $\Box B$ on the right, then by hypothesis $\Phi(\Box B, \mathsf{H}) = \mathsf{F}$. Yr instructs us to introduce a tableau t′, with $t\,R\,t'$ and B on the right of t′. But since $\Phi(\Box B, \mathsf{H}) = \mathsf{F}$, by definition there exists H' such that $\mathsf{H}\,\boldsymbol{R}\,\mathsf{H}'$, and $\Phi(A, \mathsf{H}') = \mathsf{F}$; then in the $(n+1)$th stage we can extend α by $\alpha(\mathrm{t}') = \mathsf{H}'$, and the extended α will satisfy all requirements. Finally, for Yl, if $\Box B$ appears on the left of t, then at the $(n+1)$th stage we are to put B on the left of each tableau t′ such that $t\,R\,t'$. Correspondingly in the model Φ, we have by hypothesis of the induction that $\Phi(\Box B, \mathsf{H}) = \mathsf{T}$; hence, for every H' s.t. $\mathsf{H}\,\boldsymbol{R}\,\mathsf{H}'$, we have $\Phi(B, \mathsf{H}') = \mathsf{T}$. Now by hypothesis of the induction, if $t\,R\,t'$, and $\alpha(\mathrm{t}') = \mathsf{H}'$, we have $\mathsf{H}\,\boldsymbol{R}\,\mathsf{H}'$, and hence $\Phi(B, \mathsf{H}') = \mathsf{T}$. So when we put B on the left of t′ in the $(n+1)$th stage, the requirements on α are still satisfied. Finally, in addition to the rules, the stipulations on R (reflexivity, transitivity, etc.) can lead us to assert, for certain pairs tableaux t and t′, that $t\,R\,t'$; we need to verify that correspondingly $\mathsf{H}\,\boldsymbol{R}\,\mathsf{H}'(\mathsf{H} = \alpha(\mathrm{t}), \mathsf{H}' = \alpha(\mathrm{t}'))$. This verification is immediate, since the stipulations on the relation R between tableaux are the same as the restrictions on the relation $\boldsymbol{R}$ of the m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$.

So the italicized assertion has been verified. Now since the construction is closed, there is a stage in which every alternative set contains a tableau with some formula on both left and right. By the italicized assertion, this stage contains a set $\mathscr{S}$ and a map α related to the m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ and to the model Φ in the manner described by the italicized property. Now $\mathscr{S}$ contains a tableau t with a formula B on both left and right. Hence if $\mathsf{H} = \alpha(\mathrm{t})$, since B occurs on both the left and right of t, we have $\Phi(B, \mathsf{H}) = \mathsf{T} = \mathsf{F}$, a contradiction. So the *reductio* is complete. Q.E.D.

Lemma 2. *If the construction for A is not closed, then A is not valid.*

Proof. Suppose the construction for A is not closed; then at every stage of the construction, one of the alternative sets of the stage is not closed. We intend, as in [1], to deduce from this fact the existence of a countermodel Φ to A on a m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$. This deduction is not quite so straightforward as might appear from the proof of the corresponding lemma in [1] (Lemma 2); actually the proof in [1] of that lemma was inadequate. In fact, the assertion in [1], p. 6, that "there exists a set of tableaux, one of the construction's alternative sets, which is not closed," was quite meaningless; we are guaranteed a non-closed alternative set at every stage of the construction, but there is no such thing as an "alternative set" for the whole construction.

Let us then proceed more cautiously: We notice that the $(n+1)$th stage of a construction is obtained from the nth by the application of some rule. Let $\mathscr{S}$ be an alternative set which is not closed and which us unaffected by the rule; then it appears unchanged in the $(n+1)$th stage, and we say that the set $\mathscr{S}$ of the $(n+1)$th stage is an *immediate descendant* of the set $\mathscr{S}$ of the nth stage. On the other hand, if a rule is applied to $\mathscr{S}$ in the nth stage, $\mathscr{S}$ is transformed by the rule intoo a set $\mathscr{S}'$ (or, if the rule is Λr, into two alternative sets $\mathscr{S}'$ and $\mathscr{S}''$); then the set $\mathscr{S}'$ (or sets $\mathscr{S}'$ and $\mathscr{S}''$) of the $(n+1)$th stage is (are) called an ( ) immediate descendant(s) of $\mathscr{S}$ in the nth stage. Similarly, speak of a tableau t′ in the

$(n+1)$th stage as an "immediate descendant" of a non-closed tableau t in the nth stage, under either of the following conditions: a) t is unchanged by the rule applied to obtain the $(n+1)$th stage from the nth, and t′ is the same as t; or b) t is transformed by the rule in question into t′, or (in the case of Λr) into two tableaux, one of which is t′. Both for tableaux and for alternative sets, we shall use the term "descendant" as the ancestral of the relation "immediate descendant".

Notice that, in a construction for A, we begin with only one alternative set. If then we diagram the relation "immediate descendant" (between alternative sets), we get a natural tree structure; and in fact this relation (more strictly, its converse) is easily verified to be a tree relation in the sense of 2.3. We notice that, if an alternative set $\mathscr{S}$ is closed, it has no immediate descendants, since no further rules are applied to it[1]); hence the tree corresponding to a closed construction is finite. If a construction is not closed, the tree it may be finite or infinite. Suppose it is finite; then clearly, the construction has only finitely many stages. Since the construction is not closed, the terminal stage of the construction contains at least one alternative set which is not closed. We choose such an alternative set, and call it $\mathscr{S}_0$.

Now in this finite case, it is easy to define a countermodel to A. Let $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ be a model structure in which K is the alternative ordered set $\mathscr{S}_0$, $\boldsymbol{R}$ is the relation R which orders $\mathscr{S}_0$, and G is the main tableau of $\mathscr{S}_0$. Define a model $\Phi(P, \mathsf{H})$ (P atomic, $\mathsf{H} \in \mathsf{K}$), by $\Phi(P, \mathsf{H}) = \mathsf{T}$ iff P appears on the left side of H (remember, $\mathsf{K} = \mathscr{S}_0$, a set of tableaux!); otherwise, $\Phi(P, \mathsf{H}) = \mathsf{F}$. Now we show, by induction on the number of symbols in a formula B, that if B appears on the left (right) of H, $\Phi(B, \mathsf{H}) = \mathsf{T}(\mathsf{F})$. For atomic B, appearing on the left, this is a matter of definition. If B appears on the right, and is atomic, we notice that since the construction is not closed B cannot appear on the left, and hence $\Phi(B, \mathsf{H}) = \mathsf{F}$. If $B \wedge C$ appears on the left of H, by Λl (and the fact that the construction has terminated the stage containing $\mathscr{S}_0$, so all the rules of the construction have been applied), both B and C must appear on the left of H; hence if we assume the hypothesis of the induction, $\Phi(B, \mathsf{H}) = \Phi(C, \mathsf{H}) = \mathsf{T}$, by definition it follows that $\Phi(B \wedge C, \mathsf{H}) = \mathsf{T}$. Similarly if $B \wedge C$ occurs on the right of H, either B or C does, say B; then by the inductive hypothesis, $\Phi(B, \mathsf{H}) = \mathsf{F}$, and hence $\Phi(B \wedge C, \mathsf{H}) = \mathsf{F}$, as required. The treatment of negation is similar. If $\Box B$ appears on the left of H, then by Yl (and the fact that the construction has terminated), B appears on the left of every tableau H' of K s.t. $\mathsf{H}\, \boldsymbol{R}\, \mathsf{H}'$. Hence by the inductive hypothesis, for all H' s.t. $\mathsf{H}\, \boldsymbol{R}\, \mathsf{H}'$, we have $\Phi(B, \mathsf{H}') = \mathsf{T}$; so, by definition, $\Phi(\Box B, \mathsf{H}') = \mathsf{T}$. If $\Box B$ is on the right, Yr guarantees an H' with B on the right, and $\mathsf{H}\, \boldsymbol{R}\, \mathsf{H}'$; by the inductive hypothesis, $\Phi(B, \mathsf{H}') = \mathsf{F}$, so $\Phi(\Box B, \mathsf{H}) = \mathsf{F}$, and the induction is complete. Now we need only observe that A occurs on the right of G to obtain

[1]) More explicitly, we notice that the definitions already excluded the possibility that a closed set (tableau) should have an immediate descendant; and the fact that no rule is applied to such a set (tableau) justifies this procedure. A *non-closed* set (tableau) to which no rules are applied at a stage has an immediate descendant at the next stage; namely, itself.

the conclusion that $\Phi(A, \mathsf{G}) = \mathsf{F}$, i.e., that Φ is a countermodel to A. This completes the finite case.

On the other hand, if the construction is infinite and hence not closed, we need to apply KÖNIG's *Unendlichkeitslemma* to the corresponding tree. According to this lemma, the tree, being infinite, must contain an infinite path; so, corresponding to the path, we get an infinite sequence $\mathscr{S}_1, \mathscr{S}_2, \ldots$ of non-closed alternative sets, each of which is an immediate descendant of its predecessor in the sequence. (Here $\mathscr{S}_1$ corresponds to the first stage of the construction, $\mathscr{S}_2$ to the second, etc.) Call this infinite sequence α. Notice that any t of $\mathscr{S}_n$ possesses a unique immediate descendant t' in $\mathscr{S}_{n+1}$. If t is a tableau of $\mathscr{S}_n$ which is not the immediate descendant of any tableau of $\mathscr{S}_{n-1}$, then either $n = 1$ or t was introduced into $\mathscr{S}_n$ by Yr; in either case, we call t an initial tableau of α. A sequence of tableaux, whose first term is an initial tableau of α, say a tableau of $\mathscr{S}_n$, such that each term after the first is an immediate descendant, in the sequence α, of its predecessor, is called a *pseudo-tableau* of α. The (unique) pseudo-tableau of α whose first term is the tableau which started out the construction (with A on the right) is called the *main pseudo-tableau* of α. A pseudo-tableau can contain at most one term which is a tableau of $\mathscr{S}_m$; if it contains one, we say that it has a *representative* in $\mathscr{S}_m$. If τ_1 and τ_2 are two pseudo-tableaux of α, we say that $\tau_1 \varrho \tau_2$ iff there exists an $\mathscr{S}_m$, with representatives t_1 and t_2 of τ_1 and τ_2 in $\mathscr{S}_m$, such that $\mathrm{t}_1 R \mathrm{t}_2$. Clearly, since R is reflexive, and since every pseudo-tableau contains a representative in some $\mathscr{S}_m$, ϱ is reflexive also. Notice further that if $m > n$ and τ is a pseudo-tableau with a representative in $\mathscr{S}_n$, τ has a representative in $\mathscr{S}_m$ also. Using this fact, it is easy to see that if R is transitive (symmetric), ϱ is transitive (symmetric) also.

Given a pseudo-tableau τ: $\mathrm{t}, \mathrm{t}', \mathrm{t}'', \ldots$, define a formula B as occurring on the left (right) of τ iff it occurs on the left (right) of some tableau which occurs as a term of the sequence τ. We now see that pseudo-tableaux behave like tableaux. Is fact, if $B \wedge C$ occurs on the left of τ, so do B and C, and if $B \wedge C$ occurs on the right of τ, either B or C must occur on the right. For if $B \wedge C$ occurs on the left of τ, it occurs on the left of some tableau t of the sequence τ, and hence occurs on the left of all tableaux succeeding t. At some point in the sequence a rule Λl was applied to a tableau t' containing $B \wedge C$ on the left, so that in the immediately succeeding tableau t'' B and C both appear on the left of t''. Hence, by definition, B and C appear on the left of τ. The proof that τ has similar properties for Λr, Nl, Nr goes just the same way. Further, for Yr, we can prove that if $\Box B$ appears on the right of τ, there exists a τ' in α s.t. B occurs on the right of τ', and $\tau \varrho \tau'$. For let t be a tableau of τ with B on the right; then by Yr there exists a tableau t', the initial term of some sequence (pseudo-tableau) τ', such that $\mathrm{t} R \mathrm{t}'$ and with B on the right of t'; then by definition, B appears on the right of τ', and $\tau \varrho \tau'$. Similarly, if $\Box B$ occurs on the left of τ, B occurs on the left of any tableau τ' such that $\tau \varrho \tau'$. Finally, observe that τ can contain no formula B on both left and right. For if B did occur on both sides of the sequence τ, we would have terms t and t' such that B occurs on the left of t and on the right of t'. Of the pair t and t', one must occur earlier in the sequence τ; suppose (without loss of generality) that it is t. Then B

occurs onot only on the left of t, but on the left of t', so that t', as a tableau, is closed. But since a closed tableau has no immediate descendants, the infinite sequence τ cannot contain a closed tableau, contrary to what has just been established. So no formula can appear on both sides of τ.

It should by now be evident that the set of all pseudo-tableaux τ of α, ordered by the relation ϱ, can replace the set $\mathscr{S}_0$ which was used in the preceding proof; the pseudo-tableaux have properties quite similar to the tableaux themselves. In fact, the countermodel to A now reads as follows: We define a m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ by taking G to be the main pseudo-tableau of α, K to be the set of all pseudo-tableaux τ of α, and $\boldsymbol{R}$ to be the relation ϱ. Further, define the model $\Phi(P, \mathsf{H})$ as assigning T to P iff P appears on the left of H; otherwise F. Then, as in the finite case, it is easy to show that we have a countermodel to A.

3.3. Trees and a reformulation of the rules

Each of the ordered alternative sets in a given stage of a construction has, in a natural and obvious fashion, the structure of a tree. In fact, let K be an alternative set of some stage of a construction, and let G be the descendant (in K of) the main tableau. For $t_1, t_2 \in \mathsf{K}$, we say that $t_1 \, S \, t_2$ iff at some stage of the construction there is a tableau t_1' with a formula $\Box A$ on the right, and with t_2' just introduced by Yr at this stage with A on the right, and such that t_1 and t_2 are descendants of t_1' and t_2', respectively. It is easily verified that $(\mathsf{G}, \mathsf{K}, S)$ is a tree. Further, we notice that the conditions we have imposed on the relation R can now be restated thus: R is to be the smallest relation between tableaux that contains S and satisfies the appropriate reflexivity, transitivity, and symmetry conditions. This, in turn, is precisely to say that $(\mathsf{G}, \mathsf{K}, R)$ is a tree m.s. (of the appropriate modal system) generated by the tree $(\mathsf{G}, \mathsf{K}, S)$. Now in the preceding section, it was shown that if K is a non-closed alternative set of the terminal stage of a (finite) construction for A, a countermodel to A can be associated with the m.s. $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ ($\mathsf{G} =$ main tableau of K, $\boldsymbol{R} =$ ordering relation R). The present considerations show that this is a tree model. Similar considerations (left to the reader) show that the models given in 3.2 for non-terminating constructions (in terms of pseudo-tableaux), are tree models also.

These considerations suggest that the rules, which we have stated in terms of R, could instead be stated in terms of the basic tree relation S defined in the preceeding paragraph (letting R drop out of the picture altogether). This is so. In a construction in terms of S, the rules Nl, Nr, Λl are unaltered. Λr is unaltered except that:

(1) "R" is replaced by "S" throughout (and "R_1" by "S_1", "R_2" by "S_2");

(2) the italicized condition, included to ensure reflexivity, is dropped. (The relation S, of course, is not reflexive.)

Yr also has its original form, except that "R" is replaced by "S". Yl gets the brunt of the alterations. Since the relation S is not reflexive, and is neither transitive nor symmetric even if R is, we must put in substitutes for these conditions on R into Yl. Consider first M, where R is only reflexive. This corresponds to the fact that if $\Box A$ is true in a world, so is A. We put this into the Yl of M thus:

Yl. Let $\Box A$ appear on the left of t_1. Then put A on the left of t_1 and of any tableaux t_2 such that $t_1\, S\, t_2$.

In the previous formulation, it was superfluous to require that A be put on the left of t_1, since R was reflexive. This holds no longer. For S4, we need further to obtain a transitivity surrogate. This could be handled by replacing "S" by "S^*" in the rule Yl as it was just previously stated; but we prefer a different procedure. It is based on the fact, easily verified for S4 models Φ, that if $\Phi(\Box A, \mathsf{H}) = \mathsf{T}$ and $\mathsf{H}\, \boldsymbol{R}\, \mathsf{H}'$, then $\Phi(\Box A, \mathsf{H}') = \mathsf{T}$. So we stipulate:

Yl. If $\Box A$ appears on the left of a tableau t_1, put A on the left of t_1 and put $\Box A$ on the left of any tableau t_2 such that $t_1\, S\, t_2$.

Notice that, by a further application of Yl, since $\Box A$ goes on the left of t_2, so does A. Also if later a tableau t_3 is introduced with $t_2\, S\, t_3$, since $\Box A$ appears on the left of t_2, Yl requires us to put $\Box A$ on the left of t_3. So, clearly, we have the *effect* of transitivity.[1])

The revised Yl for the BROUWERsche system follows:

Yl. Let $\Box A$ appear on the left of t_1. Then: (1) put A on the left of t_1; (2) put A on the left of every tableau t_2 such that $t_1\, S\, t_2$; (3) put A on the left of the (unique) tableau t_3 such that $t_3\, S\, t_1$, if such a tableau exists.

For an S5 construction, the rules Yl for S4 and the BROUWERsche system are combined.

Constructions in terms of S have the following useful property: if we call tableaux t_1 and t_2 "contiguous" if $t_1\, S\, t_2$ or $t_2\, S\, t_1$, then a rule applied to a tableau t can affect at most t and tableaux contiguous to t.

The present "S-formulation" of the rules will be exploited in section 4; its equivalence with the previous "R-formulation" is clear, so a detailed proof (if any is desired) is left to the reader. S-formulations are solely a matter of convenience, compared with the more basic R-formulation; if we had so desired, we could have based all proofs on R-formulations. For S5 tableaux, yet another way of formulating the rules is given in [1].

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1) Query: could an analogous device be used to get rid of *symmetry*? HINTIKKA, in his original formulation of S5, proposed the condition stated for S4, plus in essence the following additional clause: If $\Box A$ is on the right of t, and $t\, S\, t'$, then put $\Box A$ on the right of t'. On this rule, the construction for $\sim \Box A \supset \Box \sim \Box A$ is closed. But the construction for the BROUWERsche axiom, or for $\sim A \supset \Box \sim \Box A$, is not closed, so the conjectured formulation of S5 fails. In fact, this formulation lacks a GENTZEN *Hauptsatz*. On the other hand, the new formulation of S4 comes (replacing GENTZEN sequents by tableaux) from CURRY [10]; and the lack of any analogous formulation for S5 corresponds to the lack of a simple *sequenzen* formulation with *Hauptsatz* of S5.

4. Completeness theorem

4.1. Consistency property

We need to verify that every provable formula of (M, S4, S5, the BROUWERsche system) is valid in the appropriate model theory. In every case this is an easy mechanical task, especially if it is carried out with the aid of tableaux; we need only verify that every axiom is valid for the appropriate model theory, and that the rules preserve validity.

One remark is in order. It is easy to show that R1 preserves validity; for if $\Phi(A, \mathsf{H}) = \mathsf{T}$, $\Phi(A \supset B, \mathsf{H}) = \mathsf{T}$, then by the valuation rules for "$\supset$", $\Phi(B, \mathsf{H}) = \mathsf{T}$ also. But it is by no means easy to give a proof, without using the results of 3.2, that if the construction for A is closed and the construction for $A \supset B$ is closed, so is the construction for B. Such a proof would involve a tableau analogue of the GENTZEN Hauptsatz. This Hauptsatz would take the form: Let a tableau construction be given, and let $\mathfrak{t}$ be a tableau occurring in the nth stage of the construction, and let us be given two new "pseudo-constructions," in which A is added, at the nth stage to the left and right of $\mathfrak{t}$, respectively (and the rules are applied to A later on). The GENTZEN Hauptsatz asserts that if the two "pseudo-constructions" are closed, so is the original. This Hauptsatz is easily proved by model-theoretic methods, once we invoke 3.2 to assure ourselves of the equivalence of tableaux to models; but if we wish to avoid this theory, we can use a GENTZEN-like induction. In the case of quantified modality, this would allow us to prove *constructively* that the construction for A is closed iff A is provable; while the semantical proof of this result would be non-constructive. The present semantical proofs, in the domain of the propositional calculus, are either constructive or can be made such; so that an inductive proof of the GENTZEN Hauptsatz, though still interesting, is theoretically unnecessary at this stage. One might appeal, for such a proof, to the previously published GENTZEN modal systems of CURRY [10], KANGER [12], and OHNISHI and MATSUMOTO [11].

4.2. Completeness property

We can show that every valid formula A is provable by showing that if the construction for A is closed, then A is provable in the appropriate system. Here we find it convenient to invoke tableau constructions based on the relation S, rather than using the relation R.

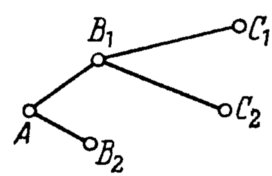
First, for a tree of tableaux ordered by the relation S at a given stage of a construction, we define the *rank* of a tableau as follows: A tableau $\mathfrak{t}$ has rank 0 in the tree if there is no tableau $\mathfrak{t}'$ s.t. $\mathfrak{t} \, S \, \mathfrak{t}'$. Otherwise, let $\mathfrak{t}_1, \ldots, \mathfrak{t}_n$ be all the tableaux $\mathfrak{t}_i$ such that $\mathfrak{t} \, S \, \mathfrak{t}_i$. Then we define $\mathrm{Rank}(\mathfrak{t}) = \mathrm{Max}\,\{\mathrm{Rank}(\mathfrak{t}_i)\} + 1$. It is easily verified that, for any finite tree of tableaux, (such as occurs in the alternative sets of a stage of a construction), a unique rank is defined for each tableau of the tree. (The tableaux of rank 0 are the endpoints of the tree; then work backwards by induction.)

Define the *associated formula* of a tableau t at a stage to be $A_1 \wedge \cdots \wedge A_m \wedge \wedge \sim B_1 \wedge \cdots \wedge \sim B_n$, where $A_1, \ldots, A_m$ are the formulae occurring on the left of t at the given stage and $B_1, \ldots, B_n$ are the formulae occurring on the right of t at that stage.

Further, define the *characteristic formula* of a tableau t at a given stage by induction on the rank of t: If t has rank 0, the charakteristic formula is the associated formula. If $\mathrm{Rank}(t) > 0$, let $t_1, \ldots, t_n$ be all the tableaux t_i such that $t\, S\, t_i$. For each t_i, $\mathrm{Rank}(t_i) < \mathrm{Rank}(t)$, so by hypothesis, the characteristic formula of the t_i have been defined already; let B_i be the characteristic formula of t_i. Further, let A be the associated formula of t. Then the characteristic formula of t is defined as $A \wedge \Diamond B_1 \wedge \Diamond B_2 \wedge \cdots \Diamond B_n$.

By the characteristic formula of a tree (ordered set) of tableaux, we mean the characteristic formula of the main tableau[1]) of the set.

Example:



Here each node represents a tableau, with the associated formula of the tableau indicated at the node. B_2, C_1, and C_2 are associated formulae of tableaux of rank 0; B_1 is the associated formula of a tableau of rank 1; and A is the associated formula of the main tableau, which is of rank 2. The characteristic formula of the tree is $A \wedge \Diamond(B_1 \wedge \Diamond C_1 \wedge \Diamond C_2) \wedge \Diamond B_2$.

Let $D_1, \ldots, D_n$ be the characteristic formulae of the alternative sets of a system of sets at a stage. Then the characteristic formula of the system is defined as $D_1 \vee \cdots \vee D_n$.

Lemma. *If A_0 is the characteristic formula of the initial stage of a construction, and B_0 is the characteristic formula of any stage, then $\vdash A_0 \supset B_0$.*

Proof. To prove the lemma, it suffices to show that for any m the characteristic formula of the mth stage implies the characteristic formula of the $(m+1)$th stage. But the characteristic formula of the mth stage in general has the form $D_1 \vee \cdots \vee D_j \vee \cdots \vee D_n$, where the D_i are characteristic formulae of alternative sets. The characteristic formula of the $(m+1)$th stage will be either

[1]) We extend the term "main tableau" to apply to any tree of tableaux in the following manner: In any such tree, the *main tableau* is the origin of the tree. Note that in any alternative set the at any stage of a construction, the "main tableau" (origin) of the set is the (unique) descendant in that set of the original main tableau which started out the construction. In short, if Ar is applied to a tableau t, then if t is auxiliary, the resulting tableaux t_1 and t_2 are auxiliary tableaux of two alternative sets; otherwise they are main tableaux of two alternative sets. And any other rule applied to a main (auxiliary) tableau leaves it main (auxiliary).

$D_1 \vee \cdots \vee D_j' \vee \cdots \vee D_n$ or $D_1 \vee \cdots \vee D_{j_1} \vee D_{j_2} \vee \cdots \vee D_n$, where the rule which obtains the $(m+1)$th stage from the mth operates solely on the alternative set whose characteristic formula is D_j. If this rule is not Λr, it will change the alternative set so that the characteristic formula be comes D_j'; but if it is Λr, the alternative set in question will "split" into two alternative sets, with respective characteristic formulae D_{j_1} and D_{j_2}. It clearly suffices, in order to prove that the characteristic formula of the mth stage implies the characteristic formula of the $(m+1)$th, to show that $D_j \supset D_j'$ or $D_j \supset (D_{j_1} \vee D_{j_2})$, as the case may be. In other words, when a rule is applied to obtain the $(m+1)$th stage from the mth, we need only consider the characteristic formula of the set to which the rule is applied. Let the formula D_j have the form

$$B \wedge \Diamond (C_1 \wedge \Diamond (E_1 \wedge \Diamond (\cdots))) \wedge \Diamond (C_2 \wedge \Diamond (E_2 \wedge \Diamond (\cdots))).$$

Now let X and Y be formulae such that $\vdash X \supset Y$ is provable. Then by R2, so is $\vdash \Box (X \supset Y)$. But in all the modal systems we have considered, (which all contain M), $\vdash \Box (X \supset Y) .\supset. \Diamond X \supset \Diamond Y$, so we have $\vdash \Diamond X \supset \Diamond Y$; we also know that $\vdash X \supset Y$ implies $\vdash X \wedge Z .\supset. Y \wedge Z$.

These two facts makes possible a great simplification of the proof that $\vdash D_j \supset D_j'$ (or $\vdash D_j \supset D_{j_1} \vee D_{j_2}$); namely we need only consider, in general, associated formulae of tableaux to which the rule we use actually applies. Suppose, for example, the rule transforms the formula C_1 into C_1' and affects no other part of D_j (i.e., D_j' is obtained from D_j by replacing C_1 by C_1' and leaving the rest unchanged.). Then if we can show $\vdash C_1 \supset C_1'$, it follows that

$$\vdash C_1 \wedge \Diamond (E_1 \wedge \Diamond (\ldots)) \supset C_1' \wedge \Diamond (E_1 \wedge \Diamond (\ldots)),$$

and hence, attaching the possibility signs, that

$$\vdash \Diamond (C_1 \wedge \Diamond (E_1 \wedge \Diamond (\ldots))) \supset \Diamond (C_1' \wedge \Diamond (E_1 \wedge \Diamond (\ldots))).$$

Now, finally, we can attach the other two conjuncts (B and $(C_1 \wedge \Diamond (E_2 \wedge \Diamond (\ldots))))$, thus obtaining $D_j \supset D_j'$. Similarly, in every case where a rule is applied to obtain the $(m+1)$th stage from the mth, we can work with only part of the "nested" characteristic formula D_j in order to obtain $D_j \supset D_j'$. Bearing these observations in mind, we break the proof down into cases, depending on the rule applied to obtain the $(m+1)$th stage from the mth (I would advise the reader to compare this proof with the analogous lemma in [1]):

Case Nl. Justified by $\vdash \sim A \supset \sim A$.

Case Nr. Justified by $\vdash \sim\sim A \supset A$.

Case Λl. Justified by $\vdash A \wedge B \supset A \wedge B$.

Case Λr. A somewhat messy case, although it is conceptually quite clear. We must prove $D_j \supset D_{j_1} \vee D_{j_2}$. Now the rule Λr applies to some formula $A \wedge B$ on the right of some tableau; therefore $\sim(A \wedge B)$ occurs as a conjunct of the associated formula of the tableau, so that $\sim(A \wedge B)$ occurs as a conjunct in the characteristic formula of the tableau. So let $E \wedge \sim(A \wedge B)$ be the characteristic formula of the tableau t in which $A \wedge B$ appears on the right; then we certainly have

$$E \wedge \sim(A \wedge B) :\supset: E \wedge \sim(A \wedge B) \wedge \sim A .\vee. E \wedge \sim(A \wedge B) \wedge \sim B.$$

If the tableau containing $A \wedge B$ on the right is the main tableau, this is the desired result $\vdash D_j \supset D_{j_1} \vee D_{j_2}$, since the characteristic formula of the main tableau of a set is the characteristic formula of the whole set. But in general we shall not be so fortunate; the tableau in question is merely auxiliary. Well, then we first observe that

$$\Diamond(E \wedge \sim(A \wedge B)) :\supset: \Diamond(E \wedge \sim(A \wedge B) \wedge \sim A .\vee. E \wedge \sim(A \wedge B) \wedge \sim B),$$

and that since $\vdash \Diamond(X \vee Y) \supset. \Diamond X \vee \Diamond Y$, we obtain

$$\vdash \Diamond(E \wedge \sim(A \wedge B)) \supset \Diamond(E \wedge \sim(A \wedge B) \wedge \sim A) \vee \Diamond(E \wedge \sim(A \wedge B) \wedge \sim B).$$

Now if t′ is the (uniquely determined) predecessor of t (the tableau t′ s.t. t′ S t), then the characteristic formula of t′ has the form $X \wedge \Diamond(E \wedge \sim(A \wedge B))$, where X is the characteristic formula t′ would have if t were removed from the tree. Using the previous results and the distributive law for "$\wedge$" over "$\vee$", we get easily

$$\vdash X \wedge \Diamond(E \wedge \sim(A \wedge B)) :\supset: X \wedge \Diamond(E \wedge \sim(A \wedge B) \wedge \sim A) .\vee.$$
$$X \wedge \Diamond(E \wedge \sim(A \wedge B) \wedge \sim B).$$

If t′ is the main tableau, we are done; otherwise we continue in the same meanner. Eventually, after sufficient labor (using each time the distributive laws $\Diamond(X \vee Y) \supset \supset \Diamond X \vee \Diamond Y$ and $(X \vee Y) \wedge Z :\supset: X \wedge Z .\vee. Y \wedge Z)$, we are driven back along the branch leading to t until we finally reach the main tableau of the tree, and obtain the desired result.

Case Y1. The rule is applied to a tableau t with $\Box A$ on the left, and with characteristic formula $\Box A \wedge X \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots$, where the E_i are the characteristic formulae of tableaux t_i with t S t_i, and $\Box A \wedge X$ is the associated formula of t.

First assume that we are dealing with a construction, based on S, of M-tableaux. In this case, we need only justify putting A on the left of t and of all the t_i such that t S t_i; the characteristic formula of t after this is done becomes $A \wedge \Box A \wedge X \wedge \wedge \Diamond(E_1 \wedge A) \wedge \Diamond(E_2 \wedge A) \ldots$ But clearly this can be obtained from the old characteristic formula, using theorems $\vdash \Box A \supset A$, and $\vdash (\Box A \wedge \Diamond E) \supset \Diamond(E \wedge A)$. If we are dealing with an S4-construction, we must justify further putting $\Box A$ on the left of each t_i such that t S t_i; but this follows, analogously, from $\vdash \Box A \wedge \Diamond E .\supset. .\supset. \Diamond(E \wedge \Box A)$, which is easily proved in S4. If we are dealing with a BROUWERsche construction, we must justify, in addition to what was done in M, putting A on the left of every tableau t′ such that t′ S t. In fact, such a tableau t′, if it exists, is unique, and its characteristic formula has the form $Y \wedge \Diamond(\Box A \wedge X \wedge \Diamond E_1 \wedge \Diamond E_2 \wedge \ldots)$.

We must show that the formula implies the new characteristic formula of t' at the $(m+1)$th stage, viz.:

$$A \wedge Y \wedge \Diamond(A \wedge \Box A \wedge X \wedge \Diamond(E_1 \wedge A) \wedge \Diamond(E_2 \wedge A) \wedge \ldots).$$

The only novelty beyond M is the occurrence of A which begins the formula. It can be obtained as follows: clearly $\vdash Y \wedge \Diamond(\Box A \wedge X \wedge \Diamond E_1 \wedge \ldots) \supset \Diamond \Box A$; but by the BROUWERsche axiom, or rather its dual, easily provable in the BROUWERsche system, we have $\vdash \Diamond \Box A \supset A$, which yields the desired result. The proof for S5 follows readily from the preceding, since the S5 procedure is just the combination of those for S4 and the BROUWERsche system, and all the preceding proofs go through in S5. (An alternative method, for S5, is of course that of [1].)

Case Yr. If $\Box A$ appears on the right of a tableau t, the tableau t has characteristic formula $X \wedge \sim \Box A$. We are instructed to start out a new tableau t' with A on the right; the characteristic formula of t becomes $X \wedge \sim \Box A \wedge \Diamond \sim A$. But clearly, since $\vdash \sim \Box A \supset \Diamond \sim A$, we also have $\vdash X \wedge \sim \Box A \supset X \wedge \sim \Box A \wedge \Diamond \sim A$. (Remark: of course, we may be required, by Yl, to put some formulae on the left of t′, immediately afterward, but this has been justified under case Yl.)

The proof is complete.

Theorem. *If A is valid* (*in* M, S4, S5, BROUWERSCHE), *it is provable* (*in the appropriate system*).

Proof. We will prove that if the construction for A is closed, then A is provable. Now since the construction for A is closed, there is a stage, say the mth, when every alternative set is closed; call the characteristic formula of this stage $D_1 \vee \cdots \vee D_m$. Let D_j be any disjunct; it is the characteristic formula of an alternative set $\mathscr{S}_j$. By hypothesis, $\mathscr{S}_j$ contains a closed tableau, whose associated formula is $X \wedge C \wedge \sim C$, where C is the formula occurring on both left and right; this formula is clearly refutable in all systems considered. In fact, so is $\Diamond(X \wedge C \wedge \sim C)$, and hence so is any formula of the form $Y \wedge \Diamond(X \wedge C \wedge \sim C)$, or even $\Diamond(Y \wedge \Diamond(X \wedge C \wedge \sim C))$. In short, using the fact that, if $\vdash \sim X$, $\vdash \sim(Y \wedge X)$ and even $\vdash \sim \Diamond(Y \wedge X)$ (the latter since by R2, $\vdash \Box \sim(Y \wedge X)$), we can obtain $\vdash \sim D_j$. Hence, since j was arbitrary, $\vdash \sim(D_1 \vee \cdots \vee D_m)$. Finally, by the lemma, the characteristic formula of the first stage of the construction implies that of the mth. But here the characteristic formula of the first stage is just $\sim A$. So the lemma tells us that $\vdash \sim A \supset D_1 \vee \cdots \vee D_m$. Since $\vdash \sim(D_1 \vee \cdots \vee D_m)$, we obtain $\vdash A$. Q.E.D.

Remark. The theorem showed, in a purely syntactical constructive manner (without reference to models), that if the construction for A is closed, A is provable in the appropriate formal system. The development of the characteristic formulae of successive stages is, in effect, a HERBRAND development from the characteristic formula of the initial stage; and we could, if we wished, base a proof procedure for modal logic solely on the development of characteristic formulae.

5. Applications

5.1. Decidability

Although in the preceding we took account of the possibility of infinite tableau constructions, it is clear that we can show that the systems are decidable (in, say, a construction of the S type) if we can show that the procedure always terminates, either in a closed system of sets or in a finite countermodel to the formula whose validity is being tested. In fact, for M and BROUWERsche constructions of the S type, we can argue as follows: Familiarly, the *modal degree* of a formula A is defined inductively thus: An atomic formula has modal degree 0; $\deg(A \wedge B) = \max(\deg(A), \deg(B))$, $\deg(\sim A) = \deg(A)$, and $\deg(\Box A) = \deg(A) + 1$. (Thus the degree is the number of "nested" necessity signs in a formula.) Finally, if t is a tableau (at a given stage of a construction), define $\deg(\mathrm{t})$ as the maximum of the degrees of the formulae occurring in t at that stage.

Using this notion of degree, we show that M and the BROUWERsche system are decidable. For let A be any formula, say first of M; we show, by induction on the degree of A, that the construction for A terminates. If the degree is 0, then A is a purely truth-functional formula, and the construction obviously terminates. Suppose the theorem has been proved for degree $\leqq m$; let A be a formula of degree $m + 1$. Then the construction for A begins with a main tableau t with A on the right. We apply, before using Yr and introducing new tableaux, all the other rules (including Yl) within the tableau t. It is easy to see that only finitely many formulae are introduces in this manner, and that these rules are exhausted in a finite number of steps. Assume we have reached the end of these steps. Then by this time the tableau t has been in general replaced, on account of applications of Λr, by various alternative tableaux $\mathrm{t}_1, \ldots, \mathrm{t}_n$ (since Yr has not yet been applied, the alternative sets at this stage are all one-element sets). Let t' be any one of these; concentrate on it. Let $\Box B_1, \ldots, \Box B_l$ be all the formulae of the form $\Box B$ appearing on the right of t'. Apply Yr, obtaining various tableaux t'_i with B_i on the right, and with $\mathrm{t}' S \mathrm{t}'_i$. Now since, $\Box B$ is a subformula of A, $\deg(\Box B) \leqq m + 1$, $\deg(B) \leqq m$. Further, if any formula $\Box C$ appears on the left of t', we put C on the left of t_i; but, by the same argument, $\deg(C) \leqq m$. It is clear that no other rules introduce into t_i formulae of degree $> m$ unless such formulae have already been introduced; so we conlude that, at every stage of the construction, $\deg(\mathrm{t}'_i) \leqq m$. In fact, in the system M, aside from putting such formulae C on the left of t'_i and a formula B_i on the right, the tableau t'_i romains entirely unaffected by the tableau t thereafter; and all the rules that are applied to or affect the tableau t have already been applied. So, in M, we can now proceed to continue the construction from each of the tableaux t'_i $(i + 1, \ldots, l)$. Notice that no tableau related to t'_i can affect any tableau related to t'_j $(i \neq j)$; this includes the statement that these tableaux do not affect each other. Hence, if we concentrate our attention on t'_i, we note that the part of the construction continuing outward from t'_i (i.e., the construction restricted to the subtree determined by t'_i) is unaffected by t'_j and t, and thus proceeds exactly as if t'_i were the main tableau of the construction. Since $\deg(t'_i) \leqq m$, the inductive hypothesis assures us that this part of the construction terminates in finitely many steps. Since there are only l tableaux t'_i, the part of the construction restricted to the tableau proceeding from t' also terminates. But further t' was an arbitrary element of a finite list of alternative tableaux; so, even if we consider all of these, we still obtain a construction that terminates in a finite number of steps.

The preceding reasoning applied to M; but it is extendable to the BROUWERsche system. Here again we have t' and l tableaux t'_i s.t. $t' S t'_i$; but in this case, there is no guarantee that t' will be unaffected by the t'_i. In fact, if a formula $\Box D$ appears on the left of t'_i, we must put D on the left of t'. Now, at least initially, when t_i is started out by putting B_i on the left (by Yr) and various formulae C_i on the right (by Yl), the argument given above shows that these formulae are of degree $\leqq m$. This property is preserved by applications of the rules; every subformula of a formula of degree $\leqq m$ is of degree $\leqq m$. Assume for the moment that $\Box D$ is a subformula of B_i or of one of the formulae of the form C_j; then $\deg(\Box D) \leqq m$, hence $\deg(D) \leqq m - 1$. Now, by applying the rules to D on the left of t', we may get a formula $\Box E_1$, which must therefore be a subformula of D, on the right of t'.[1] When we apply Yr to it, we obtain a tableau t'_{l+1} (say) started out with E_1 on the right. Notice that $\deg(\Box E_1) \leqq \deg(D) \leqq m - 1$, hence $\deg(E_1) \leqq m - 2$. Further if a subformula $\Box F$ of D appears on the left of t', we must put F on the left of every tableau t'' s.t. $t' S t''$; in particular, on the left of $t'_1, \ldots, t'_l$ and of t'_{l+1}. Notice, however, that $\deg(F) \leqq m - 2$, by the same arguments as before. Further, of course, D need not be the only formula put on the left of t by Yl at this stage; but there are only finitely many such, and all have the properties assumed for D. So we can assume that say p new tableaux $t'_{l+1}, \ldots, t'_{l+p}$ have been added, where t'_{l+i} $(i = 1, \ldots, p)$ has a formula E_i on the right, with $\deg(E_i) \leqq m - 2$. Further new formulae F are added on the left of $t'_1, \ldots, t'_l$ and $t'_{l+1}, \ldots, t'_{l+p}$, with $\deg(F) \leqq m - 2$. We can apply the rules to E_i and the F's, perhaps obtaining some subformula $\Box G$ on the left of t'_i $(i = 1, \ldots, l + p)$; but here $\Box G$ is a subformula of E_i or an F, hence $\deg(\Box G) \leqq m - 2$, so $\deg(G) \leqq m - 3$. So the new formulae G which we put on the left of t' have degree $\leqq m - 3$. Now we argue as we did before; new tableaux are started out, formulae are put on left and right of them, but now they all have degree $\leqq m - 4$. Since the degrees involved decrease by 2 at each stage, this iteration process cannot continue indefinitely. Eventually, we shall stop, obtaining a situation where we have a tableau t', tableaux $t'_l, \ldots, t'_s$ with $t' S t'_i$ $(i = 1, \ldots, s)$, and where at no later stage can the t'_i affect t' or each other. Then we can argue from here precisely as we did for the system M in the preceding paragraph.

In fact, the argument we have given guarantees a stronger result, both for M and for the BROUWERsche system: *Let A be a formula of degree m. Then the con-*

[1]) The introduction of D on the left of t' may also lead to applications of Λr to t', thus splitting t' up into several new alternative tableaux. To simplify the present proof, we ignore this (clearly inessential) possibility.

struction for A terminates. In fact, it either is closed, and hence A is valid, or it yields a finite tree countermodel for A, in which each branch of the tree is of length $\leq m$.[1]) *Hence every formula is either provable or has a finite tree countermodel.* — This fact is easily proved, by induction on m, using the methods of the preceding paragraphs.

This result does not hold for the system S4. In fact, the preceding arguments all break down in S4, since Yl in S4 allows one to transfer $\Box A$ rather than just A, when $\Box A$ occurs on the left. In fact, as K. J. J. HINTIKKA essentially pointed out to me, the formula $\sim \Box(A \wedge \Diamond \sim A .\vee. \sim A \wedge \Diamond A)$ is not valid in S4, but it has no finite tree countermodel. (Try out the tableau procedure for it!) An equivalent example, which I have used in other connections, is $\sim \Box(\Diamond A \wedge \Diamond \sim A)$. A counterexample to this would be a necessarily contingent formula. But in a finite tree model, no formula is evaluated as contingent on the endpoints of the branches of the tree (in fact on such endpoints, every true formula is necessary). Hence in S4, there are no *necessarily* contingent formulae on finite tree models.

Thus in S4 we need another argument to show the decidability. In fact, a rather traditional GENTZEN subformula argument will suffice. We will show that, although the construction can be infinite, nevertheless there is a certain stage at which the construction either closes, or terminates in a finite tree countermodel, or it is recognizable that the construction never will close. In fact, given a construction for a formula A, at every stage of the construction, in each of the tableaux of the alternative sets in that stage only subformulae of A ean occur. Now the subformulae of A are finite in number. Further, if we regard a tableau as an ordered pair of sets of subformulae of A, it is clear that there are only finitely many tableaux (i.e., if A has n subformulae, there are 2^n sets of subformulae, hence 2^{2n} pairs of such sets). Now suppose we are given a construction for A which is infinite (and therefore, not closed). Then, as in the proof of Lemma 2, section 3.2, there exists an infinite sequence $\mathscr{S}_1, \mathscr{S}_2, \ldots$ of non-closed alternative sets, each of which (except $\mathscr{S}_1$, which corresponds to the first stage of the construction) is an immediate descendant of its predecessor. Since, as was shown above, there are only finitely many distinct tableaux which can occur in a construction for A, it follows that there must be some $\mathscr{S}_j$ which is *saturated*; i.e., it has the property that for $k > j$, all the tableaux occurring in $\mathscr{S}_k$ are *equal* to tableaux already occurring in $\mathscr{S}_j$. (Here two tableaux are *equal* iff they contain the same formulae on both left and right; they need not be, however, *identical*, since they may have different positions in the tree structure.) Now suppose the construction is subjected to a modification defined as follows: We say that a tableau $\mathfrak{t}'$ *contains* a tableau $\mathfrak{t}$ iff every formula appearing on the left (right) of $\mathfrak{t}$ appears on the left (right) of $\mathfrak{t}'$.

Let a tableau $\mathfrak{t}$ be introduced at the nth stage of a construction by Yr in an alternative set $\mathscr{S}$. We call $\mathfrak{t}$ *repetitive* iff there exists an $m < n$, and a tableau $\mathfrak{t}'$ in a set $\mathscr{S}'$ at the mth stage of the construction, such that $\mathfrak{t}'$ contains $\mathfrak{t}$, and such that $\mathfrak{t}'$ still contains $\mathfrak{t}$ at later stages, as long as no rules are applied to formulae occurring within $\mathfrak{t}$ (although $\mathfrak{t}$ may get additional formulae on the left as a result of Yl applied

[1]) A branch with $i + 1$ points is said to be of length i.

to some other tableau). It is clear that if t is repetitive at the nth stage, this fact can be recognized (using an effective test) at that stage; further it is clear that the rules subsequently applied to t will merely duplicate (be "mirror images of") rules applied to t'. (In particular if a tableau t_1 is introduced by applying Yr to t, with t S t_1, at some stage of the construction a tableau t_1' is introduced, containing t_1, s.t. t' S t_1'.) Hence, in order to see whether the construction closes, it is unnecessary to apply rules to t; anything that "happens to" t will be duplicated by something that happens to t'. Hence we place the following restriction on all constructions: *No rule is to be applied to a formula occuring in a repetitive tableau.*

The preceding discussion makes it clear that when the constructions are modified so as to conform to this restriction, they must terminate. For otherwise, as was shown above, there would be an infinite sequence $\mathscr{S}_1, \mathscr{S}_2, \ldots$ of non-closed alternative sets; and this sequence would contain a "saturated" term $\mathscr{S}_j$. But by the definition of "saturated", for all $k > j$ every tableau in $\mathscr{S}_k$ is either the descendant of some tableau in $\mathscr{S}_j$ or is repetitive. Now there are only finitely many tableaux, say n in $\mathscr{S}_j$; then exactly n of the tableaux in any $\mathscr{S}_k$ $(k > j)$ are descendants of tableaux in $\mathscr{S}_j$. Further, each of the n tableaux in $\mathscr{S}_j$ contains only finitely many formulae; and in fact, in any one tableau t there are only finitely many rules that can be applied to formulae in t until all formulae are decomposed into atomic components. Hence under our new reduction the sequence $\mathscr{S}_1, \mathscr{S}_2, \ldots$ actually terminates at some $\mathscr{S}_h$ $(h \leqq j)$, contrary to the hypothesis that it is infinite. Hence, by *reductio*, the construction must be finite. This shows that the decision problem for S4 is solvable also.

Suppose we are dealing with an S4 construction, which is not closed, for a formula A. Then it still, however, must terminate; and the last stage of the construction must contain a non-closed alternative set $\mathscr{S}$. Now $\mathscr{S}$ contains only finitely many tableaux. We wish to obtain, from the set $\mathscr{S}$, a countermodel to the formula A, by the techniques of Lemma 2, section 3.2; but since the rules have not been applied to the repetitive tableaux of $\mathscr{S}$, the argument of that lemma no longer applies. This difficulty can be solved by "identifying" each repetitive tableau with a corresponding non-repetitive tableau. To put the matter more exactly: the non-repetitive tableaux determine equivalence classes of tableaux as follows: every non-repetitive tableau is assigned to its own equivalence class. A repetitive tableau t is contained in an earlier tableau t'; this tableau t' may itself be repetitive, but in this case it in turn is contained in an earlier tableau t'', etc. Eventually we find a non-repetitive tableau t_0 containing t; we place t in the equivalence class determined by t_0. (The choice of t_0 need not be unique; but once it has been made, it places t in a unique equivalence class.) Then we take the set K to consist of all the equivalence classes of tableaux thus determined; G is the equivalence class containing the main tableau of the set. Given two equivalence classes H_1 and H_2, in K, we say that $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_2$ iff there exists tableaux t_1 and t_2 in H_1 and H_2, respectively, with $t_1 R t_2$. Finally, if P is any propositional variable, and $\mathsf{H} \in \mathsf{K}$, define $\Phi(P, \mathsf{H}) = \mathsf{T}$ iff there exists a tableau t in the equivalence class H with P on the left of t; otherwise $\Phi(P, \mathsf{H}) = \mathsf{F}$. Then it is easy to show, by the method of Lemma 2, section 3.2, that $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ is a model structure, and that Φ is a countermodel to the formula A.

It is clear that the preceding paragraph has shown that every formula A of S4 is either derivable or has a finite countermodel. This result is weaker than the corresponding results for M and the BROUWERsche system, since the finite countermodel need not be a tree model. On the other hand, the proof of Lemma 2, section 3.2, guarantees that non-derivable formula always has a *countable tree* countermodel. For some formulae, like $\sim \Box (A \wedge \Diamond \sim A \,.\!\vee.\, \sim A \wedge \Diamond A)$, either the finiteness or the tree structure must be sacrificed; we need not sacrifice both. In fact (as we shall illustrate by an example but not prove), the modified version of construction, which disallows applications of rules to repetitive tableaux, gives us a constructive method for obtaining denumerable tree countermodels as well as finite countermodels. For example, to find a countermodel to the formula

$$\sim \Box (A \wedge \Diamond \sim A \,.\!\vee.\, \sim A \wedge \Diamond A),$$

we apply a tableau construction (with the restriction on repetitive tableaux) which at its last stage has an alternative set of the following form (abbreviating $A \wedge \Diamond \sim A \,.\!\vee.\, \sim A \wedge \Diamond A$ by $X_1 \vee X_2$):[1])

$$\begin{array}{c|c} \Box(X_1 \vee X_2) & \sim \Box(X_1 \vee X_2) \\ X_1 \vee X_2 & \\ X_1 & \\ A & \\ \Diamond \sim A & \\ \end{array}_{\,\mathrm{t}_1} \rightarrow \begin{array}{c|c} \sim A & A \\ \Box(X_1 \vee X_2) & \\ X_1 \vee X_2 & \\ X_2 & \\ \Diamond A & \\ \end{array}_{\,\mathrm{t}_2} \rightarrow \begin{array}{c|c} A & \\ \Box(X_1 \vee X_2) & \\ X_1 \vee X_2 & \\ X_1 & \\ \Diamond \sim A & \\ \end{array}_{\,\mathrm{t}_3}$$

[1]) Here, if we wished, we could imagine "$\Diamond$" replaced by its definition "$\sim \Box \sim$". But we have carried out the procedure as if the rules Zl and Zr, dual to Yl and Yr and essentially (using the definition of "$\Diamond$") derived from them, were assumed:
Zl. If $\Diamond A$ appears on the left of a tableau t, start out a tableau t′, with A on the left, and t R t′.
Zr. If $\Diamond A$ appears on the right of t, put A on the right of every tableau t′ s.t. t R t′.

One of the beauties of tableau procedures is that independent rules can be given for any number of connectives so that the connectives are "independent" of each other. Notice that, if we are using both Y rules and Z rules in the same system, the relations R in both must of course be the same in order to derive such theorems as $\sim \Box \sim A \supset \Diamond A$. We could introduce two relation R and R', R for the Y rules and R' for the Z rules, and then $\sim \Box \sim A \supset \Diamond A$ would no longer hold. (In fact, it is easy to introduce a model theory for several $\Box$ operators and several $\Diamond$ operators, each with its own associated relation R; the different operators can even satisfy different reduction axioms, corresponding to different properties of these relations.) This fact has apparently been overlooked in OHNISHI and MATSUMOTO [11]. For both M and S4, their rules for necessity and possibility work in isolation but not when they are combined; in neither S4 nor M (as formulated by the combined rules) is $\sim \Box \sim A \rightarrow \Diamond A$ provable. The revised rules for necessity in S4 (when both nedessity and possibility are primitive) should read:

$$\mathrm{Yl}\,\frac{\Gamma, A \rightarrow \Gamma'}{\Gamma, \Box A \rightarrow \Gamma'} \qquad\qquad \mathrm{Yr}\,\frac{\Box\Gamma \rightarrow A, \Diamond\Gamma'}{\Box\Gamma \rightarrow \Box A, \Diamond\Gamma'}$$

Similarly for Zl and Zr; and similarly for M. (The S5 rules in [11], lacking the *Hauptsatz*, are less interesting; cf. footnote 5.)

Here the tableau t_3 is repetitive, being contained in t_1. Now we can use the tableau diagram to obtain a finite countermodel, as in the preceding paragraph. In this case, we "identify" t_1 and t_3, placing them in a single equivalence class G. Thus we have a model structure, consisting of a set K with two elements G and H, with $\boldsymbol{R}$ holding between any ordered pair of elements of K. A model Φ is defined on the model structure $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ by taking A as a propositional variable and stipulating that $\Phi(A, \mathsf{G}) = \mathsf{T}$ and $\Phi(A, \mathsf{H}) = \mathsf{F}$. This is clearly a finite countermodel to $\sim \Box (A \wedge \Diamond \sim A \,.\!\vee\!.\, \sim A \wedge \Diamond A)$. On the other hand, it is equally possible to interpret the diagram so that it yields a tree countermodel. Let us take a countable set K with elements H_i indexed on the positive integers. "G" is another name for H_1, and $\mathsf{H}_i \,\boldsymbol{R}\, \mathsf{H}_j$ iff $i \leqq j$. On the model structure $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ we define a model Φ by $\Phi(A, \mathsf{H}) = \mathsf{T}$ if n is odd, and $\Phi(A, \mathsf{H}) = \mathsf{F}$ if n is even. Then Φ is a denumerable tree countermodel to $\sim \Box (A \wedge \Diamond \sim A \,.\!\vee\!.\, \sim A \wedge \Diamond A)$. Further, it is easy to see how this model was obtained from the diagram. Instead of *identifying* t_1 and t_3, we noticed that in a construction without restriction on repetitive tableaux, the same rules would apply to t_3 as to t_1, thus producing a tableau t_4 like t_2, which in turn gives rise to a tableau t_5 like t_1, etc. To every such tableau t_n we correspond a world H_n, thus obtaining the countermodel in question.

Analogous arguments yield the result that every non-derivable formula of S5 has a finite countermodel, and that S5 is decidable; essentially this result is already in [1]. The finite countermodels for S5 are automatically tree countermodels, so S5 enjoys the strong property of M and the Brouwersche system that every formula is either derivable or has a finite tree countermodel.

The decision procedures may have seemed onerous from their description, but a little practice will verify that they are no more tedious than the usual decision methods for classical propositional logic, and that they are the simplest decision procedures in the literature for modal logics.

5.2. Matrices, finite and infinite

It is usual in propositional calculus to obtain independence and non-derivability results by the use of finite many-valued truth tables. Prior, in his [14], has given us a method of interpreting such matrices in terms of possible worlds. We apply and extend his method to the present semantical analysis.

Given a model structure $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$, we define a *proposition* (or perhaps more accurately, *modal value of a proposition*), as a mapping whose domain is K and whose range is the set $\{\mathsf{T}, \mathsf{F}\}$. (Intuitively, a proposition is something that can be *true* or *false* in each world; and, for our present purposes, we identify propositions that are *strictly equivalent*, i.e., have the same truth-value in each world. Without the identification of strictly equivalent propositions, the term "modal value of a proposition" would be better). Notice that each proposition determines a unique set of worlds (the set of all worlds mapped into T), and that conversely each set of worlds determines a proposition (its "characteristic function"). Thus a proposition could just as well have been defined simply as a subset of K.

Now a *model* is a function Φ, mapping ordered pairs (P, H) into truth values, where P is a propositional variable and $\mathsf{H} \in \mathsf{K}$. For fixed P, the model Φ determines a singulary function $\lambda\, \mathsf{H}\, \Phi(P, \mathsf{H})$, mapping elements of K into truth-values; i.e., it determines a proposition. Thus a model can be viewed as a function mapping formal propositional variables into propositions.

If ϱ and σ are propositions (i.e., for $\mathsf{H} \in \mathsf{K}$, $\varrho(\mathsf{H})$ and $\sigma(\mathsf{H})$ are truth-values), we define $\varrho \wedge \sigma$ as the proposition such that $\varrho \wedge \sigma(\mathsf{H}) = \mathsf{T}$ iff $\varrho(\mathsf{H}) = \sigma(\mathsf{H}) = \mathsf{T}$; otherwise $\varrho \wedge \sigma(\mathsf{H}) = \mathsf{F}$. Further, $\sim\varrho$ is defined by $\sim\varrho(\mathsf{H}) = \mathsf{T}$ iff $\varrho(\mathsf{H}) = \mathsf{F}$, otherwise $\sim\varrho(\mathsf{H}) = \mathsf{F}$. (More compactly, assuming $\wedge$ and $\sim$ defined on truth-values in the obvious manner, $(\varrho \wedge \sigma)(\mathsf{H}) = \varrho(\mathsf{H}) \wedge \sigma(\mathsf{H})$, $(\sim\varrho)(\mathsf{H}) = \sim(\varrho(\mathsf{H}))$. If we construe ϱ and σ alternatively as subsets of K, $\varrho \wedge \sigma$ is the intersection of ϱ and σ, while $\sim\varrho$ is the complement in K of ϱ.) Finally, define $\Box\varrho$ by $(\Box\varrho)(\mathsf{H}) = \mathsf{T}$ if $\varrho(\mathsf{H}') = \mathsf{T}$ for all H' with $\mathsf{H}\, \boldsymbol{R}\, \mathsf{H}'$; otherwise $(\Box\varrho)(\mathsf{H}) = \mathsf{F}$. Notice then that if a model Φ maps the formulae A and B into the propositions ϱ and σ, then by the previous definition given for models it maps $A \wedge B$ into $\varrho \wedge \sigma$, $\sim A$ into $\sim\varrho$, and $\Box A$ into $\Box\varrho$.

Now if K has finitely many elements, say n, then there are exactly 2^n propositions in the model structure $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$. We can label these by integers $1, 2, \ldots, 2^n$, and set up "many-valued truth tables" for the operations $\wedge$, $\sim$, and $\Box$, (indicating for each propositions ϱ and σ what propositions $\varrho \wedge \sigma$, $\sim\varrho$, and $\Box\varrho$ are). If $\varrho(\mathsf{G}) = \mathsf{T}$, we call ϱ a "designated value." For example if $\mathsf{K} = \{\mathsf{G}, \mathsf{H}\}$, viewing propositions as subsets of K let $1 = \{\mathsf{G}, \mathsf{H}\}$, $2 = \{\mathsf{G}\}$, $3 = \{\mathsf{H}\}$, $4 =$ empty set. Then 1 and 2 are designated. If $\boldsymbol{R}$ is reflexive and $\mathsf{G}\, \boldsymbol{R}\, \mathsf{H}$ but not $\mathsf{H}\, \boldsymbol{R}\, \mathsf{G}$, the matrix thus obtained is Group II of [4]; if $\boldsymbol{R}$ relates every ordered pair of elements of K, the resulting truth-table is Group III of [4]. In the first case, since $\boldsymbol{R}$ is transitive but not symmetric, the matrix satisfies S4 but not S5; in the second case, where transitivity and symmetry hold, the matrix satisfies S5.

Generally, then corresponding to each finite model structure $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ we get a finite many-valued matrix which satisfies the appropriate modal system. Further, this matrix is *normal* in the sense of Church [20]. Now in the preceding section it was shown that every non-derivable formula of each of our modal systems has a finite countermodel. If we translate this countermodel into a normal matrix for the modal system, we see that every formula is either derivable or has a non-designated value in some finite normal matrix satisfying the axioms of the system. This property has been called the "finite model property" (cf. [21]).

Of course the restriction to finite sets K is inessential to the construction we have outlined; if K has n elements, finite or infinite, the propositions of $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ form a matrix of 2^n elements.

Now consider a countable set K with a tree relation $\boldsymbol{S}$; the relation $\boldsymbol{S}$ is to be constructed so that for each $\mathsf{H} \in \mathsf{K}$, there are denumerably many $\mathsf{H}' \in \mathsf{K}$ s.t. $\mathsf{H}\, \boldsymbol{S}\, \mathsf{H}'$. Then let $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ be the tree model (for the appropriate modal system) generated by the relation $\boldsymbol{S}$. Then it can be shown that $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ is a "universal" model structure in the following sense: any non-derivable formula A has a countermodel Φ in $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$. In fact, for M, S5, and the BROUWERsche system we have shown that every non-derivable formula has a finite tree countermodel. Let the finite tree model structure of this countermodel be $(\mathsf{G}', \mathsf{K}', \boldsymbol{R}')$, let $\boldsymbol{S}'$ be the tree relation generating it, and let Φ' be the countermodel to A. We define a function χ, mapping K into K' (intuitively, "identifying" elements of K with their images in K') thus: $\chi(\mathsf{G}) = \mathsf{G}'$. If $\chi(\mathsf{H}) = \mathsf{H}'$ has already been defined, let H_i $(i = 0, 1, 2, \ldots)$ be the countably many elements of K such that $\mathsf{H}\, \boldsymbol{S}\, \mathsf{H}_i$, and let $\mathsf{H}_1', \ldots, \mathsf{H}_n'$ be the finitely many elements ($n = 0$ if there are none) of K' such that $\mathsf{H}'\, \boldsymbol{S}\, \mathsf{H}_i'$. Define $\chi(\mathsf{H}_0) = \mathsf{H}'$, $\chi(\mathsf{H}_i) = \mathsf{H}_i'$ $(1 \leq i \leq n)$, and $\chi(\mathsf{H}_i) = \mathsf{H}'$ $(i > n)$. Then χ has been inductively defined; and we define a countermodel Φ to A in $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ by $\Phi(P, \mathsf{H}) = \Phi'(P, \chi(\mathsf{H}))$. Then Φ is a countermodel to A if Φ' is.

A method similar to that outlined in the previous section for obtaining repetitive countermodels suffices to show that an S4 model structure generated by the tree of the preceding paragraph is "universal" for S4. (The proof will not be given here.)

Now the set K of the universal model structure is denumerable; and hence it contains continuously many propositions; so each of the four modal systems has a characteristic matrix of the cardinality of the continuum. In fact, however, we need only include in the matrix those propositions used for countermodels. Thus, e.g., for all systems except S4, we need only consider those propositions which are of the form $\varrho(\mathsf{H}) = \varrho'(\chi(\mathsf{H}))$, where ϱ' is a proposition defined on a finite model structure, and χ is the mapping of the preceding paragraph. Since there are only denumerably many propositions of this form (and the same fact can be verified for S4), it follows that each of the systems considered has a denumerable characteristic matrix.

The characteristic matrices, either of the cardinality of the continuum or the denumerable ones, just defined are all normal. Hence none of the systems we consider can be unreasonable in the sense of HALLDÉN [16], since HALLDÉN's paper shows essentially that no system with his "bad" property can possess a normal characteristic matrix.

One additional comment: In [1], an extended notion of "two-valued truth-table", based on the model-theoretic considerations, was introduced for S5. Essentially the corresponding notions for M and S4 were given by ANDERSON [13] (except that the reductions to normal form appear unnecessary).

5.3. A property of M and S4

As an example of the power of the present semantical techniques, we derive the following property of M and S4, previously known from algebraic arguments of McKINSEY-TARSKI [15] and LEMMON [18]: *If* $\Box A \vee \Box B$ *is derivable, then either* A *or* B *is derivable (and hence, by* R2, *either* $\Box A$ *or* $\Box B$ *is derivable).* For suppose neither A nor B derivable; then let Φ and Φ' be countermodels to A and B, defined on the model structures $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ and $(\mathsf{G}, \mathsf{K}', \boldsymbol{R}')$ respectively. Clearly we can assume that K and K' are disjoint. Define a model structure $(\mathsf{G}'', \mathsf{K}'', \boldsymbol{R}'')$, where $\mathsf{G}'' \notin \mathsf{K}$, $\mathsf{G}'' \notin \mathsf{K}'$, $\mathsf{K}'' = \mathsf{K} \cup \mathsf{K}' \cup \{\mathsf{G}''\}$, and for $\mathsf{H}_1, \mathsf{H}_2 \in \mathsf{K}''$, $\mathsf{H}_1 \boldsymbol{R}'' \mathsf{H}_2$ iff either a) $\mathsf{H}_1, \mathsf{H}_2 \in \mathsf{K}$ and $\mathsf{H}_1 \boldsymbol{R} \mathsf{H}_2$ or b) $\mathsf{H}_1, \mathsf{H}_2 \in \mathsf{K}'$ and $\mathsf{H}_1 \boldsymbol{R}' \mathsf{H}_2$, or c) $\mathsf{H}_1 = \mathsf{G}''$. Then $\boldsymbol{R}''$ is reflexive, and is transitive if $\boldsymbol{R}$ and $\boldsymbol{R}'$ are (using the disjointness of K and K'). (Note that this statement would not hold for symmetry.) So $(\mathsf{G}'', \mathsf{K}'', \boldsymbol{R}'')$ is a model structure for the appropriate system. Let Φ'' be any model in $(\mathsf{G}'', \mathsf{K}'', \boldsymbol{R}'')$ such that $\Phi''(P, \mathsf{H}) = \Phi(P, \mathsf{H})$ for $\mathsf{H} \in \mathsf{K}$, and $\Phi''(P, \mathsf{H}) = \Phi'(P, \mathsf{H})$ if $\mathsf{H} \in \mathsf{K}'$. (Since K and K' are disjoint, there exist maps Φ'' satisfying these conditions.) Now we verify by induction that $\mathsf{H} \in \mathsf{K}$, $\Phi''(C, \mathsf{H}) = \Phi(C, \mathsf{H})$ for every formula C. For if C is atomic then this is part of the definition of Φ''. If $C = D \wedge E$ or $C = \sim D$, the inductive step is easy. If $C = \Box D$, and the statement has been verified for D, then if $\Phi(D, \mathsf{H}') = \mathsf{T}$ for all H' with $\mathsf{H} \boldsymbol{R} \mathsf{H}'$, then $\Phi(C, \mathsf{H}) = \mathsf{T}$; otherwise $\Phi(C, \mathsf{H}) = \mathsf{F}$. But $\Phi(D, \mathsf{H}') = \Phi''(D, \mathsf{H}')$ by hypothesis. If $\mathsf{H} \in \mathsf{K}$, conditions b) and c) cannot hold (with $\mathsf{H} = \mathsf{H}_1$, $\mathsf{H}' = \mathsf{H}_2$), so that $\mathsf{H} \boldsymbol{R}'' \mathsf{H}'$ if and only if $\mathsf{H} \boldsymbol{R} \mathsf{H}'$. So $\Phi(C, \mathsf{H}) = \mathsf{T}$ iff $\Phi''(C, \mathsf{H}) = \mathsf{T}$; i.e., $\Phi(C, \mathsf{H}) = \Phi''(C, \mathsf{H})$. Q.E.D. Similarly, if $\mathsf{H} \in \mathsf{K}'$, $\Phi''(C, \mathsf{H}) = \Phi'(C, \mathsf{H})$. Hence in particular, since Φ and Φ' are countermodels for A and B, respectively, $\Phi''(A, \mathsf{G}) = \Phi(A, \mathsf{G}) = \mathsf{F}$, $\Phi''(B, \mathsf{G}') = \Phi'(B, \mathsf{G}') = \mathsf{F}$. Since $\mathsf{G}'' \boldsymbol{R}'' \mathsf{G}$, we have hence $\Phi''(\Box A, \mathsf{G}) = \mathsf{F}$; similarly since $\mathsf{G}'' \boldsymbol{R} \mathsf{G}'$, $\Phi''(\Box B, \mathsf{G}'') = \mathsf{F}$. Hence $\Phi''(\Box A \vee \Box B, \mathsf{G}'') = \mathsf{F}$, and Φ'' is a countermodel to $\Box A \vee \Box B$. So if neither A nor B is provable, $\Box A \vee \Box B$ is not provable; so the desired result follows.

The result fails for S5 and the Brouwersche system. In fact, in both systems the Brouwersche axiom $A \supset \Box \Diamond A$ holds. Putting $\Diamond B$ for A, we get $\Diamond B \supset \Box \Diamond \Diamond B$. Equivalently $\Box \sim B \vee \Box \Diamond \Diamond B$ is derivable, but clearly neither $\sim B$ nor $\Diamond \Diamond B$ is derivable.

6. Other systems

Consideration of various non-normal systems is reserved for another paper. Prior's Q (cf. [14]), and similarly constructed modifications of the systems considered in this paper, are better motivated by a consideration of quantification theory, and hence are reserved for a paper on quantified modal systems. An example of a system between S4 and S5 is the System S4.3 of [21], obtainable by adding the scheme $\Diamond A \wedge \Diamond B \,.\!\supset\!.\, \Diamond(A \wedge \Diamond B) \vee \Diamond(B \wedge \Diamond A)$ to S4. Model-theoretically, this amounts to a requirement on a S4-model-structure $(\mathsf{G}, \mathsf{K}, \boldsymbol{R})$ that if $\mathsf{H}, \mathsf{H}' \in \mathsf{K}$, then either $\mathsf{H} \boldsymbol{R} \mathsf{H}'$ or $\mathsf{H}' \boldsymbol{R} \mathsf{H}$. Other systems formed by imposing various requirements on $\boldsymbol{R}$ can easily be constructed by consideration of [11], Theorem 3.5.

If we were to drop the condition that $\boldsymbol{R}$ be reflexive, this would be equivalent to abandoning the modal axiom $\Box A \supset A$. In this way we could obtain systems of the type required for deontic logic.

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