Decision Procedures

A modal formula χ is in S5 iff it's true in every world in the canonical S5 frame whose atomic sentences are the ones appear in the sentence. If there are n atomic sentences, there will be $2^{2^n + n - 1}$ worlds in the canonical frame, so we can test whether χ is in S5 by going through the worlds one by one to see if we find one where χ is false.

S5 is exceptional in having finitely many worlds in the canonical frame. If we try to apply the same procedure to another normal modal system, S4 for instance, we'll find that, to determine whether $\Box\, \varphi$ is true in a world, we have to look at infinitely many other worlds to see whether φ is true in them. So we are confronted with an infinite task in determining where a sentence is true in a particular world.

To get a decision procedure, we look at a pared down version of the canonical frame construction. Instead of taking our worlds to be complete stories, we look at short stories that decide the truth or falsity of the subformulas of the formula χ that we are testing, but regard questions about the other formulas as irrelevant.

To get a decision procedure for the smallest normal modal system K, we'll take a world in $W^{K,\chi}$ to be a maximal K-consistent set of subformulas of χ and negations of subformulas of χ. Here a set of sentences is K-consistent if it is included within a complete story that also includes K. We are treating a formula as a subformula of itself. If a conjunction is a subformula of χ, it's in a world in $W^{K,\chi}$ iff both conjuncts are in the world. A disjunction that's a subformula of χ is in a world iff one or both disjuncts are in the world. A subformula of χ is in a world w iff its negation isn't in w. For α an atomic formula in χ and w a world, $I^{K,\chi}(\alpha,w)$ = True iff $\alpha \in w$. We define the accessibility relation $R^{K,\chi}$ by stipulating that w $R^{K,\chi}$ v iff, whenever $\Box\, \psi$is in w, $\psi$is in v.

We want to show that, for any world w in $W^{K,\chi}$, for any subformula of χ, the subformula is true in w iff it's an element of w. Once we've proved this, we can say that, if χ is in K, χ is in every world in every frame, so it's true in every world in $<W^{K,\chi},R^{K,\chi},I^{K,\chi}>$. If χ isn't in K, then K $\cup \{\sim \chi\}$ is truth-functionally consistent, so there is a maximal K-consistent set $@^{K,\chi}$ of subformulas of χ and negated subformulas of χ that includes $\sim \chi$. So $@^{K,\chi}$ is a world in $<W^{K,\chi},R^{K,\chi},I^{K,\chi}>$ in which χ is false. If χ has k subformulas, there are at most 2^k members of $W^{K,\chi}$. So we can test whether χ is true in every world in the model by checking them all. This gives us an algorithm for testing whether a given modal formula is in K.

The proof that a subformula of χ is true in a world iff it's an element of the world proceeds by induction on the complexity of formulas. For atomic formulas, this is immediate from the way $I^{K,\chi}$ is defined. The clauses for the SC connectives goes the same as always. We have to worry about formulas of the form $\Box\, \varphi$.

If $\Box\, \varphi$ is an element of w, φ is an element of every world accessible from w. So by inductive hypothesis, φ is true in every world accessible from w. So $\Box\, \varphi$ is true in w.

If $\Box\, \varphi$ is true in w, φ is true in every world accessible from w. By inductive hypostheis, φ

is an element of every world accessible from w. So there isn't any maximal K-consistent set of subformulass of χ and negations of subformulas of χ that includes all the formulas ψ with $\Box\psi$ an element or w and excludes φ. So there isn't any K-consistent set of formulas that includes all the formulas ψ with $\Box\psi$ in w and excludes φ. In other words, if $\Box\psi_1, \Box\psi_2, \ldots, \Box\psi_k$ are the subformulas of χ that begin with "$\Box$," the conditional $(\psi_1 \rightarrow (\psi_2 \rightarrow \ldots (\psi_k \rightarrow \varphi)\ldots))$ is an element of K. Using (Nec) and (K), we conclude that $(\Box\psi_1 \rightarrow (\Box\psi_2 \rightarrow \ldots (\Box\psi_k \rightarrow \varphi)\ldots))$ is in K, and hence in w. Since each $\Box\psi_i$ is in w, $\Box\varphi$ is in w.

A slight modification gives us a decidion procedure for KT. Given a formula χ, define $W^{KT,\chi}$ to be the set of maximal KT-consistent subformulas of χ and negated subformulas of χ. Define $R^{KT,\chi}$ by stipulating that w $R^{KT,\chi}$ v iff, for every elemnt of w of the form $\Box\theta$, θ is in w. $I^{KT,\chi}(\alpha,w)$ = True iff $\alpha \in$ w. The argument we just gave shows that, if $\chi \notin$ KT, χ is false is some world in the frame. $R^{KT,\chi}$ is reflexive, so if $\chi \in$ KT, it's true in every world in the frame. So we can test whether χ is in KT by checking whether it's true in every world in the frame.

Given χ, define:

$W^{K4,\chi}$ = {maximal K4-consistent sets of subformulas of χ and their negations.
w $R^{K4,\chi}$ v iff. whenever $\Box\varphi$ is in w, φ and $\Box\varphi$ aire in v.
$I^{K4,\chi}(\alpha,w)$ = True iff $\alpha \in$ w.

χ is in K4 iff it's true in every world in the frame. Since $R^{K4,\chi}$ is reflexive, this gives us a decision proceduer for K4. Replacing "K4" by "KT4" gives us a decision procedure for KT4, which is Lewis's S4.

Given χ, define:

$W^{KB,\chi}$ = {maximal KB-consistent sets of subformulas of χ and their negations.
w $R^{KB,\chi}$ v iff, whenever $\Box\varphi$ is in w, φ is in v, and whenever $\sim\varphi \in$ w, $\sim\Box\varphi \in$ w.
$I^{KB,\chi}(\alpha,w)$ = True iff $\alpha \in$ w.

We can verify that, whenevery φ is a subformula of χ, φ is true in a world w iff it's an element of w, and also that $R^{KB,\chi}$ is symmetric. This gives us a decision procedure for KB. Replacing "KB" by "KTB" yields a decision procedure for KTB.

Given χ, define:

$W^{K4B,\chi}$ = {maximal K4B-consistent sets of subformulas of χ and their negations.
w $R^{KB4,\chi}$ v iff whenever $\Box\varphi$ is in w, φ and $\Box\varphi$ aire in v, and whenever either $\sim\varphi$ $\sim\Box\varphi$ is in w, $\sim\Box\varphi$ is in v.
$I^{K4B,\chi}(\alpha,w)$ = True iff $\alpha \in$ w.

We can verify that, whenevery φ is a subformula of χ, φ is true in a world w iff it's an element of w, and also that $R^{K4B,\chi}$ is symmetric and transitive. This gives us a decision procedure for K4B. We could easily upgrade this to a decision procedure for KT4B, but there's no need. KT4B is just S5.

The same technique works for K45. It doesn't quite work for K5. K5 is decidable, but to show this, we have to modify the method we've used so for by allowing extra formulas into a world beyond the subformulas of χ and the negations of subformulas of χ. The tricky part is to make sure that, once we start allowing additional formulas into a world, we don't wind up with infinite worlds. I won't go through it here. See Krister Segerberg, "Decidability of Four Modal Logics," *Theoria* 34 (1968): 21-25.