Uninterpreted Frames and Binary Relations

An uninterpreted frame is a pair $<W,R>$ where W is a nonempty set and R is a binary relation on W. A formula of a language for the modal sentential calculus is *valid for* $<W,R>$ iff it's true in every model $<W,R,I,@>$.

R is reflexive iff every formula in KT is valid for $<W,R>$.

Proof: (⇒) follows from the fact that KT is valid for every reflexive model.
(⇐) Suppose $w \in W$ and not $w\ R\ w$. Pick an atomic sentence α and define I so that $I(\alpha,v) = \text{True}$ iff $v \neq w$. α is true in every world accessible from w, so $\Box\, \alpha$ is true in w, even though α isn't treu in w. So $(\Box\, \alpha \rightarrow \alpha)$ isn't true in $<W,R,I,w>$.

R is transitive iff every formula in K4 is valid for $<W,R>$

Proof: (⇒) follows from the fact that K4 is valid for every transitive model.
(⇒) Suppose w, v, and u are in W, with $w\ R\ v$ and $v\ R\ u$, but not $w\ R\ u$. Pick α and define I so that $I(\alpha,t) = \text{True}$ iff $w\ R\ t$. $\Box\alpha$ is true in w. α isn't true in u, so $\Box\alpha$ isn't true in v, so $\Box\, \Box\, \alpha$ isn't true in w. So $(\Box\, \alpha \rightarrow \Box\, \Box\, \alpha)$ isn't true in $<W,R,I,w>$.

This follows immediately:

R is reflexive and transitive iff every formulain KT4 is valid for $<W,R>$.

R is Euclidean iff every formula in K5 is valid for $<W,R>$.

Proof: (⇒) We already know that K5 is valid for every Euclidean model.
(⇐) Suppose that w, v, and u are in W, that wRv and wRu but not vRu. Define I so that α is true in u only. Then $\Diamond\alpha$ is false in v, so $\Box\Diamond\alpha$ is false in w, even though $\Diamond\alpha$ is true in w. So $(\Diamond\alpha \rightarrow \Box\Diamond\alpha)$ is false in $<W,R,I,w>$.

We get similar results for every normal modal system obtained by adding one or more of (T), (4), (B), and (5) to K.