Normal Modal Systems

We talked before about simple Kripke models. Now things are going to get more complicated.

A *Kripke model* is an ordered quadruple <W,R,I,@>, where W, the set of *worlds*, is a nonempty set; R, the *accessibility relation* is a binary relation on W; I, the *interpretation function,* is a function that assigns to each pair <α,w>, with α an atomic sentence and w a world, either the value True or the value False; and @ ∈ W is the *actual world*. The triple <W,R,I> is a *frame*. For φ and atomic sentence and w a world, φ is true in w if and only if I(φ,w) = True. A conjunction is true in w iff both conjuncts are true in w. A disjunction is true in w if and only if one or both disjuncts are true in w. A negation is true in w iff the negatum isn't true in w. □φ is true in w iff φ is true in every world v with Rwv. A sentence is *true* in the model iff it's true in @.

A sentence is *valid* for a frame or set of frames iff it's true at every world in every member of the set.

A *normal modal system* is a set of sentences Γ with the following properties:

Every tautology is in Γ.

Every instance of schema (K) is in Γ

Γ is closed under (MP) and (Nec).

Lemma. If Γ is a normal modal system, then every tautological consequence of Γ is in Γ.

Proof: By Lindenbaum's Lemma, if χ is a tautological consequence of Γ, then there are elements $\gamma_1, \gamma_2, \ldots, \gamma_n$ of Γ such that the conditional $(\gamma_1 \rightarrow (\gamma_2 \rightarrow \ldots (\gamma_n \rightarrow \chi)\ldots))$ is a tautology. Since it's a tautology, it's in Γ. By n applications of modus ponens, χ is an Γ. ⊠

Theorem. For Γ a set of sentences, the following are equivalent:

(i) Γ is a normal model system.

(ii) There is a class of frames such that Γ is the set of formulas valid for every member of the class .

(iii) Either Γ is the set of all formulas or there is a frame $<W,R,I>$ such that Γ is the set of sentences valid for $<W,R,I>$.   .

Proof: That (ii) implies (i) is easy to check. That (iii) implies (ii) is immediate; if Γ is the set of all sentences, Γ is the set of sentences valid for every member of the empty class of frames. If Γ is the set of sentences valid for $<W,R,I>$, it's the set of sentences valid for every member of the class $\{<W,R,I>\}$. So we only need to worry about showing that (i) implies (iii).

If Γ is a truth-functionally inconsistent normal modal system, then for any sentence χ, χ is a tautological consequence of Γ. By the lemma, χ is an element of Γ. So a truth-functionally inconsistent normal modal system includes every sentence.

Now suppose that Γ is a truth-functionally consistent normal modal system. We want to define a frame, the *canonical frame* for Γ, in which all and only the members of Γ are valid.

Let W, the set of "worlds," be the set of complete stories that include Γ.

Define R by stipulating that Rwv iff, whenever $\Box\varphi$ is in w, φ is in v.

Let I set $I(\alpha,w)$ equal to True iff α is in w.

For this to be frame, W has to be nonempty, which it is because Γ is truth-functionally consistent. We establish the following:

Claim. A sentence is true in a world iff it's an element of the world.

Proof: We prove that the claim holds for every world w and formula θ by induction on the complexity of θ. So suppose the claim holds for all formulas simpler than θ. There are five cases. The first four are easy:

Case 1. θ is atomic. θ is true in w iff $I(\theta,w)$ = True iff $\theta \in w$.

Case 2. θ is a disjunction, say $(\varphi \vee \psi)$. $(\varphi \vee \psi)$ is true in w iff either φ or ψ is true in w iff either φ or ψ is an element of w [by inductive hypothesis] iff $(\varphi \vee \psi)$ is an element of w.

Case 3. θ is a conjunction, say $(\varphi \wedge \psi)$. $(\varphi \wedge \psi)$ is true in w iff both φ and ψ are true in w iff both φ and ψ are elements of w [by inductive hypothesis] iff $(\varphi \wedge \psi)$ is an element of w.

Case 4. θ is a negation, say $\sim \varphi$. $\sim \varphi$ is true in w iff φ isn't true in w iff φ isn't an element of w [by inductive hypothesis] iff $\sim \varphi$ is an element of w.

Case 5. θ has the form $\Box\varphi$.

(⇐) Suppose $\Box \varphi$ is in w. Take any world v with Rwv. Then φ is in v. By inductive hypothesis, φ is true in v. Since v was arbitrary, φ is true in every world accessible from w. So $\Box \varphi$ is true in w.

(⇒) Now suppose that $\Box\varphi$ isn't an element of w. We want to see that there is a world accessible from w in which φ isn't true. This means, according to the inductive hypothesis, that we want a world accessible from w that excludes φ. That is, given the definitions of W and R, we want a complete story that includes Γ and also includes all the sentences ζ with $\Box\zeta$ in w but that doesn't include φ. If γ is in Γ, then $\Box \gamma$ is in Γ, by (Nec), and so in w. So it will be enough to find a complete story that includes all the sentences ζ with $\Box \zeta \in w$. If there is no such complete story, then by Lindenbaum's Lemma, there are elements $\zeta_1, \zeta_2, \ldots, \zeta_n$ with each $\Box \zeta_i$ in w such that $(\zeta_1 \rightarrow (\zeta_2 \rightarrow \ldots (\zeta_n \rightarrow \varphi)\ldots))$ is a tautology. Since it's a tautology, it's in Γ, and so, by (Nec), $\Box(\zeta_1 \rightarrow (\zeta_2 \rightarrow \ldots$

$(\zeta_n \rightarrow \varphi)...))$ in in Γ. By n applications of (K), $(\Box\, \zeta_1 \rightarrow (\Box\, \zeta_2 \rightarrow ... (\Box\, \zeta_n \rightarrow \Box\, \varphi)...))$ is in Γ, and so in w. Since w is closed under modus ponens, $\Box\, \varphi$ is in w. Contradiction. ⊠

Using the claim, we want to show that, for any formula θ, θ is in Γ iff θ is valid for the canonical frame for Γ. If φ is in Γ, φ is in every complete story that includes Γ. So φ is in every world. So by the claim, is true in every world.

If φ isn't in Γ, φ isn't a tautological consequence of Γ. So there is a complete story w that includes Γ and excludes φ. w is a world, and by the claim, φ isn't true in w. ⊠

If Γ is a normal modal system, then for any formula outside Γ, there is a world in the canonical frame in which the sentence is false.

Let me write down some axioms schemata; the schemata were named by different people at different times, so the nomenclature is annoyingly haphazard:

(T) $(\Box\varphi \rightarrow \varphi)$

(4) $(\Box\varphi \rightarrow \Box\Box\varphi)$

(B) $(\varphi \rightarrow \Box\Diamond\varphi)$

(5) $(\Diamond\varphi \rightarrow \Box\Diamond\varphi)$

Let me also write down some notable properties of binary relations:

R is a *reflexive* relation on W iff, for each w in W, we have Rww.

R is *transitive* iff, for each u, v, and w, if Ruv and Rvw, then Ruw.

R is *symmetric* iff, for each u and v, if Ruv, then Rvu.

R is *Euclidean* iff, for each u, v, and w, if Ruv and Ruw, then Rvw.

K is defined to be the smallest normal modal system (that is, the normal modal system that's included in every other normal modal system), so that a sentence is an element of K iff it is derivable from instances of schema (K) by the rules TC and Necessitation. A sentence is in K iff it is true in every world in every frame. Why? The set of sentences valid for every frame is a normal modal system, so it includes K. If φ isn't in K, then there is a frame in which there is a world in which φ is false, namely, the canonical model for K.

KT is defined to be the smallest normal modal system that includes (T), so that a formula is an element of KT iff it is derivable from (K) and (T) by the rules TC and Necessitation. A formula is an element of KT iff it is true in every world in every reflexive frame. Why? Given a model <W,R,I,@>, with R reflexive, if □φ is true in @, then φ is true in every world accessible from @; in particular, φ is true in @ itself; so all instances of schema (T) are true in the model. Consequently, the set of sentences valid for every reflexive frame is a normal modal system that includes (T). Moreover, the canonical frame for KT is reflexive; for any world w in the canonical frame, if □φ is in w, φ is in w, so we have Rww. Thus, if φ isn't in KT, then there is a reflexive frame in which there is a world in which φ is false, namely, the canonical frame for KT. Thus we have:

A formula is in KT iff it's true in every reflexive model.

We've actually proved something a bit stronger: If a normal modal system includes (T), its canonical model is reflexive.

There is a connection between axiom schema (4) and transitive frames exactly analogous to the connection between (T) and reflexive frames. We have:

If R is transitive, (4) is valid for <W,R>.

Proof: Suppose □φ is true in w. Take any v accessible from w. Now take a world u accessible from v. u is accessible from w, so φ is true in u. Since φ is true in every world accessible from v,

$\Box\varphi$ is true in v. Since v was an arbitrary world accessible form w, we conclude that $\Box\varphi$ is true in every world accessible from w, so that $\Box\Box\varphi$ is true in w.

If Γ is a normal modal system that includes (4), the canonical frame for Γ is transitive.

Proof: Suppose w R v and v R u. Suppose $\Box\varphi$ is true in w. Because $(\Box\varphi \rightarrow \Box\Box\varphi)$ is in w, $\Box\Box\varphi$ is in w. So $\Box\varphi$ is in v. So φ is in u. This holds for any φ, so w R u.⊠

The collection of formulas valid for the class of transitive frames is a normal modal system that includes (4). So it includes the smallest normal modal system that includes (4), which is K4. If a sentence isn't in K4, there is a world in the canonical frame for K4 in which it is false, and the canonical frame for K4 is transitive. We have proved:

A formula is in K4 iff it's true in every transitive model

The set of formulas valid for the class of reflexive, transitive frames in a normal modal system that included both (T) and (4), so it includes the smallest normal modal system that incluludes both (T) and (4), which is KT4. If a formula isn't in KT4, there is a world in the canonical frame for KT4 in which it's false, and the canonical frame for KT4 is reflexive and transitive. We have proved:

A formula is in KT4 iff it's true in every reflexive, transitive model.

KT4 is Lewis's S4.

In a similar way, we have:

If R is symmetric, (B) is valid for <W,R>.

If Γ is a normal modal system that includes (B), the canonical frame for Γ is symmetric.

These give us:

A formula is in KB iff it's true in every symmetric model

A formula is in KTB iff it's true in every reflexive, symmetric model.

A formula is in K4B iff it's true in every transitive, symmetic model

A formula is in KT4B iff it's true in every reflexive, transivite, symmetric model

KT4B is the same as S5.

Again, we have:

If R is Euclidean, (5) is valid for <W,R>.

If Γ is a normal modal system that includes (5), the canonical frame for Γ is Euclidean.

This gives us another batch of theorems.

Historically, the core use of the modal sentential calculus was to formalize the logic of logical necessity. For that purpose, the accessibility relation has no role to play. We can treat each role as having access to itself and all the other worlds. What the introduction of accessibility relations accomplishes is to make the machinery usable for other purposes. We've seen a example: In the semantics of the "It is determined that" operator, we use the accessibility relation to show how something can be determined now that was unsettled earlier. In epistemic logic, the "worlds" are epistemic states, and w has access to v if someone in epistemic state w could get to epistemic state v by learning something new. In deontic logic, the accessibility relation is used to limit the options for what you ought to do to things that you are actually capable of doing. Accessibility plays a key role in the provability logic, where the "possible worlds" are models of a mathematical theory. In tense logic the "worlds" are instants of time and past instants have access to future instants. The introduction of the accessibility relation makes the modal machinery much more versatile, for we can adjust the apparatus to suit the varying needs of varying purposes.