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On Intuitionistic Logic[1]

AREND HEYTING*

1 Introduction

In two excellent articles, Mr. Glivenko has expounded the principles of formal intuitionistic logic (*Bulletin* 14, p. 225, and **15**, p. 183). However, in the course of the discussion begun by the article by Mr. Barzin and Mr. Errera, several questions were raised whose resolution does not depend on the composition of a formal system, since they concern the meanings of the terms used, which vary according to one's point of view. In the present article I propose, first of all, to bring more clearly to the fore the intuitionistic point of view without, moreover, elaborating on its philosophical justification; then, to examine how the logic conceived of by Mr. Glivenko is complicated by the addition of the idea of provability Mr. Levy has brought into discussion and that he unduly identifies with the Brouwerian assertion. The resolution of the last problem will help very much to achieve the goal expressed at the beginning.

The realists speak of the existence of mathematical entities, giving this word its ordinary meaning; Mr. Levy pretends that everybody understands this language (*Revue de Métaphysique et de Morale* **33** (1926), p. 547). This is a very audacious assertion, because from the moment one leaves the domain of daily life, where the exact meaning of a word has less importance than its efficacy, and enters the domain of philosophy, the meaning of the words "to exist" gives rise to a controversy of the deepest kind; it is on this point that the grand systems part. If such is the case for the notion of the existence of material objects, how much more uncertain and obscure must be the meaning of the existence of mathematical entities. Should one be astonished that Mr. Brouwer rejects such an equivocal idea as a legitimate means of mathematical proof? Here, then, is an important result of the intuitionistic critique: *The idea of an existence of mathematical entities outside our minds must not enter into the proofs.* I believe that even the realists, while continuing to believe in the transcendent [*transcendante*] existence of mathematical entities, must recognize the importance of the question of knowing how mathematics can be built up without the use of this idea.

For the intuitionists, mathematics constitutes a grandiose edifice constructed by

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human reason. Maybe they would do better to avoid completely the words "to exist"; if they continue, nevertheless, to use them, these words would have no other meaning for them than "to be constructed by reason."

2 Assertion

A proposition p like, for example, "Euler's constant is rational," expresses a problem, or better yet, a certain expectation (that of finding two integers a and b such that $C = a/b$), which can be fulfilled [*réalisée*] or disappointed [*déçue*]. The *assertion* of p has, in classical logic, the meaning "p is true"; this "classical assertion" designates a transcendent fact of nature that does not conform with the intuitionistic ideas. If, for example, a realist were to say: "It is true that C is rational," he could just as well say: "There exist two integers a and b such that $C = a/b$"; one sees that the classical assertion implies the idea of transcendent existence. One does not escape from this criticism by replacing, with Mr. Levy, "p is true" by "p is provable," since this last sentence, being equivalent to "there exists a proof of p," implies again the idea of transcendent existence. To satisfy the intuitionistic demands, the assertion must be the observation of an empirical fact, that is, of the realization of the expectation expressed by the proposition p. Here, then, is the *Brouwerian assertion* of p: *It is known how to prove* p. We will denote this by $\vdash p$. The words "to prove" must be taken in the sense of "to prove by construction." For example, one will not have proved that C is rational other than by indicating the means of calculating the integers whose quotient is C; this is not a bad definition of the word "rational"; it is the only definition possible. ("C is rational if there exist two integers whose quotient is C" would imply the notion of transcendent existence; "C is rational if it is not irrational" would upset the natural order of the ideas.)

Let us remark once more that, in classical logic as in intuitionistic logic, the assertion of a proposition is not itself a proposition, but the observation of a fact.[2] In classical logic it is a transcendent fact; in intuitionistic logic it is an empirical fact.

3 Negation

Let a proposition p be given; the classical negation "p is false" cannot be of use in intuitionistic logic, for the same reasons as for the classical assertion; it must be replaced by "p implies a contradiction." Let us denote this "Brouwerian negation" of p by $\sim p$; then $\sim p$ is a new proposition expressing the expectation of being able to reduce p to a contradiction; the negation is a logical function. $\vdash \sim p$ will mean: "it is known how to reduce p to a contradiction."

Evidently, between $\vdash p$ and $\vdash \sim p$ there is a third case, in which one knows neither how to prove p nor how to prove $\sim p$. This case could be denoted by p', but it must be realized that p' will hardly ever be a definitive statement, since it is necessary to take into account the possibility that the proof of either p or $\sim p$ might one day succeed. If one does not wish to risk having to retract what one has said, in the case p' one should not state anything at all. It is in this way that one obtains the logic devised by Mr. Glivenko, and which I have developed in more detail in a recent paper.[3]

4 Double Negation

The meaning of $\vdash\sim\sim p$ is now clear: "One knows how to reduce to a contradiction the supposition that p implies a contradiction." The case $\vdash\sim\sim p$ can occur without $\vdash p$ being fulfilled; many such examples have been given by Brouwer, Wavre, and others. Let us cite again the following:

Let π be the sequence [*suite*] of digits in the decimal expansion of the number π; π_n the sequence of the first n digits of π; S the set consisting of π and all the π_n (n runs over all of the positive integers). Finally let ρ be the sequence that is obtained by writing π, but under the condition that the sequence is terminated as soon as a "string" [*séquence*] 0123456789 is encountered. For p we take the proposition "ρ is equal to an element of S." $\vdash p$ would mean that one could indicate the element of S equal to ρ (the existence of such an element should not enter into our reasoning; for that one must know either the number of decimal digits of π after which a string occurs, or that a string will never occur; neither of these two is the case in the present state of science). However, one can assert $\vdash\sim\sim p$; here is the proof: $\sim p$ indicates the supposition that there is a contradiction if one supposes that $\rho = \pi$ or that $\rho = \pi_n$ for some n; that is to say, that there is a contradiction if one assumes that a string never occurs and also if one assumes that it occurs after a finite number of decimal digits; this is already the desired contradiction, proving $\sim\sim p$.

5 Provability

A proof of the proposition p is a mathematical construction; the expectation of being able to construct such a proof thus constitutes a new proposition we will denote by $+p$ (to be read as: "p is provable"). The formula $\vdash +p$ has exactly the same meaning as $\vdash p$; however, p does not coincide with $+p$. To prove this, let us consider the proposition "every even number is the sum of two prime numbers" (Goldbach's conjecture). Then p means simply that in taking an even number at random, one expects to be able to find two primes of which it is the sum. (This possibility is decided after a finite number of attempts.) $+p$ on the contrary requires a construction that gives us this decomposition for all even numbers at the same time. One cannot prove the first without proving the second, but the difference between the two propositions is glaring if the negations are taken. In order to be able to assert $\vdash\sim +p$, it suffices to reduce to a contradiction the supposition that one can find a construction proving p; by that one will not yet have proved that the supposition p itself implies a contradiction. If we appeal to the example of Goldbach's conjecture, we find: $\vdash\sim +p$ means that one will never be able to find a rule that effects in advance the decomposition for all even numbers; this does not mean that there is a contradiction when one supposes that in taking an even number at random, one will always be able to divide it into two prime numbers. It is even conceivable that it could one day be proved that this last supposition cannot lead to a contradiction; then one would have at the same time $\vdash\sim +p$ and $\vdash\sim\sim p$. One should abandon every hope of ever settling the question; the problem would be unresolvable.

The difference between p and $+p$ disappears if p requires a construction; this is the case for every negative proposition, for according to the definition of Section 3, such a proposition requires the construction of a contradiction. The propositions $\sim p$ and $+\sim p$ are thus equivalent.

6 Amplified Logic

Let us try to find out which combinations of the two logical functions $\sim$ and $+$ have different meanings. $+\sim p$ is identical to $\sim p$, both formulas expressing the expectation of being able to reduce p to a contradiction. $++p$ is identical to $+p$, for one would not know how to conceive of a proof of p, without thereby imagining a proof of the provability of p, and conversely. Finally, Mr. Brouwer has proven that $\sim\sim\sim p$ is identical to $\sim p$.[4] From these results we infer that each proposition that can be formed from p by the repeated application of the functions $\sim$ and $+$ is equivalent to one contained in the following table:

$$p\begin{cases} +p & \sim +p & \sim\sim +p \\ \sim p & \sim\sim p & \cdots \end{cases}$$

Consequently, every judgment on the logical value of p will be equivalent to the assertion of one or more of these propositions. One easily verifies that only seven different cases are possible.

The results 1–3 are definitive. In case 4, one could hope to pass one day to 2 or to 3; from 5 one could get to 1; from 6 to 5, 3, or 1; from 7 to all the others. At any moment in time, each proposition is definitely in one of these cases; the question of knowing in which it is to be found "in truth" (transcendently) is not a question of mathematics.

Up to now, examples are known only in cases 1, 2, 6, and 7; it does not seem likely that a proposition belonging to one of the other cases will soon be found. A logic that would treat properties of the function $+$ would therefore be purely hypothetical; in view of the task that is incumbent upon the intuitionistic mathematicians, namely, the reconstruction of all of mathematics, one cannot ask them to de-

Asserted Propositions	Consequences	Excluded Assertions	Predicate
1. $\vdash p$ (or $\vdash +p$)	$\vdash\sim\sim p$ $\vdash\sim\sim +p$	$\vdash\sim p$ $\vdash\sim +p$	Proven (true)
2. $\vdash\sim p$	$\vdash\sim +p$	All others	Contradictory (false)
3. $\vdash\sim +p$ $\vdash\sim\sim p$		All others	Unsolvable (unresolvable)
4. $\vdash\sim +p$		$\vdash +p$, $\vdash p$ $\vdash\sim\sim +p$	Unprovable
5. $\vdash\sim\sim +p$	$\vdash\sim\sim p$	$\vdash\sim p$, $\vdash\sim +p$	Not unprovable
6. $\vdash\sim\sim p$		$\vdash\sim p$	Not contradictory
7. None			Not decided (not resolved)

velop this logic. Nevertheless, by the exposition of the first elements, I hope to have shed a new light on the intuitionistic ideas.

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Notes

1. Presented by Mr. Th. De Donder.
2. In their *Principia Mathematica*, Mr. Russell and Mr. Whitehead have insisted on this difference between a proposition and its assertion.
3. Die formalen Regeln der intuitionistischen Logik. (*Sitzungsber. Pr. Akad. d. Wiss.*, Berlin, 1930.)
4. *Verse. Kon. Ak. v. Wet.* Amsterdam, XXXII, p. 877; *Jahresber. D.M.V.* **33**, p. 251.

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The Formal Rules of Intuitionistic Logic

AREND HEYTING*

Introduction

Intuitionistic mathematics is a mental activity [*Denktätigkeit*], and for it every l guage, including the formalistic one, is only a tool for communication. It is in pr ciple impossible to set up a system of formulas that would be equivalent to in itionistic mathematics, for the possibilities of thought cannot be reduced to a fin number of rules set up in advance. Because of this, the attempt to reproduce[1] t most important parts of mathematics in formal language is justified exclusively the greater conciseness and determinateness of the latter vis-à-vis ordinary languag and these are properties that facilitate the penetration into the intuitionistic concep and the use of these concepts in research.

For the construction of mathematics it is not necessary to set up logical la of general validity; these laws are discovered anew in each single case for the mat ematical system under consideration. But linguistic communication, which is stru tured in accordance with the needs of everyday life, proceeds in the form of log cal laws, which it presupposes as given. A language that were to reflect step by st the workings of intuitionistic mathematics would in all its parts deviate so mu from the usual form that it would lose entirely the favorable properties mention above. These considerations have led me to begin the formalization of intuitioni tic mathematics again with a propositional calculus.

The formulas of the formal system originate from a finite number of axio through the application of a finite number of operational rules. In addition to co stant signs, they also contain signs for variables. Now, the relationship between th system and mathematics is that under a certain interpretation of the constants a under certain restrictions concerning the substitution of variables, each formula re resents a correct mathematical assertion. (For example, the variables in proposition calculus can only be substituted by meaningful mathematical propositions.) If a sy tem is so constructed as to satisfy the condition mentioned last, then its *consisten* is also guaranteed in the sense that it cannot contain any formula that would repr sent a contradictory sentence in that interpretation.

*"Die formalen Regeln der intuitionistischen Logik," *Sitzungsberichte der Preussischen Akademie d Wissenschaften*, 1930, pp. 42–56. Translated from the German by Paolo Mancosu. Published by