SEMANTICAL ANALYSIS OF INTUITIONISTIC LOGIC I

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The present paper gives a semantical model theory for Heyting's intuitionist predicate logic, and proves the completeness of that system relative to the modelling. The model theory and completeness theorem were announced in [1]. The semantics for modal logic which we announced in [1] and developed in [2], [3], together with the known mappings of intuitionistic logic into the modal system S4, inspired the present semantics for intuitionist logic. It would in fact be possible to derive the completeness of Heyting's predicate logic in our semantics by using the mappings into S4 together with the results of [2], [3]. We prefer, however, to develop the semantics of intuitionistic logic independently of that of S4; this procedure will enable us, we believe, to obtain somewhat more information about intuitionistic logic, including the mapping into S4 as a consequence thereof[1]). Further, a fairly recently worked-out development, not contained in the announcement of [1], is included: an exposition of Cohen's notion of "forcing" [5] in terms of the present semantics. In addition to giving a simple decision procedure for Heyting's propositional calculus, Part II will present a result not announced in [1] but mentioned in [4]—the undecidability of monadic intuitionistic quantification theory. The proof is based on the semantics previously developed.

It should be mentioned that, for the pure implicational intuitionistic propositional logic, Beth [6] has announced the rediscovery of essentially the present modelling; also that, for all of intuitionist propositional logic,

[1]) The reader who wishes to understand thoroughly the deeper motivation of the present paper, however, is strongly urged to consult [2], [3], and [16], which give the underlying analysis of modal logic.

a modelling equivalent to ours can be extracted from the results of Lemmon and Dummett [7].[1])

The results of this paper, though devoted to intuitionistic logic, are proved only classically, except as mentioned below. Intuitionistically, the situation is essentially the same as that for Beth's completeness theorem [8], as analysed by Dyson and Kreisel in [9]; a reader who is interested in intuitionistically valid proofs can consult [9] and apply a similar analysis to the present results. We will give indications below which (we believe) will be sufficient for a reader familiar with [9] to make such an analysis. In the course of these indications, we will prove some results about Kreisel's system FC which are parenthetical to the main theme of this paper. In particular, we will show that Kuroda's conjecture and Markov's principle are both refutable in FC.

Some notations that will be used throughout the paper are the following: P^n, Q^n, R^n ($n \geq 0$) are n-adic predicate letters; a 0-adic predicate letter is usually called a "proposition(al) letter." Occasionally the superscript on a predicate letter will be omitted if this does not sacrifice clarity. We use letters x, y, z,..., with or without subscripts, as (individual) variables. The formulae of the intuitionistic propositional calculus are to be built out of the usual connectives $\wedge$, $\vee$, $\supset$, $\neg$, starting with the propositional letters as atomic formulae. In the predicate calculus, not only propositional letters but also formulae $P^n(x_1, \ldots, x_n)$ are taken as atomic; thence formulae are built up from these in the usual manner, using the connectives just given and the quantifiers (x) and $(\exists x)$. We use A, B, C,..., for arbitrary formulae of propositional or predicate calculus, depending on the context; if we wish to call attention to certain free variables in a formula, we use such notations as $A(x_1, \ldots, x_n)$. We assume, finally, that the reader is familiar with standard presentations of Heyting's formalized intuitionistic propositional and predicate calculus, say the presentation in [10].

[1]) Kreisel's conjectured "reinterpretation of the (intuitionistic) logical constants" in [17] is also, if his conjectures prove correct, related to the present model theory.

1. The model theory

We define an (intuitionistic) *model structure* (m. s.) to be an ordered triple $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ where $\mathbf{K}$ is a set, $\mathbf{G}$ is an element of $\mathbf{K}$, and $\boldsymbol{R}$ is a reflexive and transitive relation on $\mathbf{K}$.

An (intuitionistic) model on a m. s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is a binary function $\phi(P, \mathbf{H})$, where P ranges over arbitrary proposition letters[1]) and $\mathbf{H}$ ranges over elements of $\mathbf{K}$, whose range is the set $\{\mathbf{T}, \mathbf{F}\}$, and which satisfies the following condition: if $\phi(P, \mathbf{H}) = \mathbf{T}$ and $\mathbf{H}\boldsymbol{R}\mathbf{H}'$ $(\mathbf{H}, \mathbf{H}' \varepsilon \mathbf{K})$, then $\phi(P, \mathbf{H}') = \mathbf{T}$.

Given a model $\phi(P, \mathbf{H})$, we can define a value $\phi(A, \mathbf{H})$ $(= \mathbf{T}$ or $\mathbf{F})$ for an arbitrary formula A of propositional calculus by induction on the number of connectives in A. If A has no connectives, then it is a proposition letter and $\phi(A, \mathbf{H}) = \mathbf{T}$ or $\mathbf{F}$ has already been defined for each $\mathbf{H}$. Assume that $\phi(A, \mathbf{H})$ and $\phi(B, \mathbf{H})$ have already been defined. Then we stipulate:

a) $\phi(A \wedge B, \mathbf{H}) = \mathbf{T}$ iff $\phi(A, \mathbf{H}) = \phi(B, \mathbf{H}) = \mathbf{T}$; otherwise, $\phi(A \wedge B, \mathbf{H}) = \mathbf{F}$.

b) $\phi(A \vee B, \mathbf{H}) = \mathbf{T}$ iff $\phi(A, \mathbf{H}) = \mathbf{T}$ or $\phi(B, \mathbf{H}) = \mathbf{T}$; otherwise, $\phi(A \vee B, \mathbf{H}) = \mathbf{F}$.

c) $\phi(A \supset B, \mathbf{H}) = \mathbf{T}$ iff for all $\mathbf{H}' \in \mathbf{K}$ such that $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, $\phi(A, \mathbf{H}') = \mathbf{F}$ or $\phi(B, \mathbf{H}') = \mathbf{T}$; otherwise, $\phi(A \supset B, \mathbf{H}') = \mathbf{F}$.

d) $\phi(\neg A, \mathbf{H}) = \mathbf{T}$ iff for all $\mathbf{H}' \in \mathbf{K}$ such that $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, $\phi(A, \mathbf{H}') = \mathbf{F}$; otherwise, $\phi(\neg A, \mathbf{H}) = \mathbf{F}$.

Notice that the conditions on $\wedge$ and $\vee$ are exact analogues of the corresponding conditions on classical conjunction and disjunction; but the conditions on $\supset$ and $\neg$ are not analogous to the classical conditions. It is easy to show by induction, for any $\mathbf{H}, \mathbf{H}' \in \mathbf{K}$ such that $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, that if $\phi(A, \mathbf{H}) = \mathbf{T}$, then $\phi(A, \mathbf{H}') = \mathbf{T}$. This property has been stipulated

[1]) In [2], we let $\phi(P, \mathbf{H})$ range over $\mathbf{H} \varepsilon \mathbf{K}$ and atomic subformulae of a fixed formula A. We called this a model of A. We could equally well have adopted this orientation here; conversely [2] could have adopted, *mutatis mutandis*, the present definition. The viewpoint of [2] is exploited in the analysis of Cohen's "forcing", where we consider models assigning values only to formulae built out of a fixed atomic formula $P(x)$.

We should also remark that, although in this section we have taken the atomic formulae to be proposition letters and formulae $P^n(x_1, \ldots, x_n)$, the definitions would equally well go through if formulae were built out of an arbitrary fixed class of atomic formulae; this fact is exploited in the "provability interpretation," section 1.3, below.

for a propositional letter, and it follows for more complex formulae using the clauses $(a) - (d)$.

Notice that, intuitionistically, the inductive definition here given does not work, since it clearly appeals to the law of excluded middle in clause (c) and (d) (*e.g.*, in (d), either for all $\mathbf{H}'$, $\phi(A, \mathbf{H}') = \mathbf{F}$, or not). Thus intuitionistically, it would be best to define a model ϕ as a mapping $\phi(A, \mathbf{H})$ in $\{\mathbf{T}, \mathbf{F}\}$, where A ranges over *arbitrary* formulae of propositional calculus, and which happens to satisfy the clauses $(a) - (d)$ as well as the condition that $\phi(P, \mathbf{H}) = \mathbf{T}$ and $\mathbf{H}\boldsymbol{R}\mathbf{H}'$ implies $\phi(P, \mathbf{H}') = \mathbf{T}$. Clearly, from the classical viewpoint, this modification leaves the notion of a model essentially unchanged.

We call a formula A of propositional calculus *valid* iff $\phi(A, \mathbf{G}) = \mathbf{T}$ for every model ϕ on a model structure $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$. A model ϕ on a m. s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, such that $\phi(A, \mathbf{G}) = \mathbf{F}$, is called a *countermodel* for A.

To extend the modelling to quantification theory, we define a *quantificational model structure* (q. m. s.) to be a model structure $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, together with a function ψ (the "domain function"), defined on $\mathbf{K}$, such that $\psi(\mathbf{H})$ is a non-empty set for all $\mathbf{H} \in \mathbf{K}$, and $\psi(\mathbf{H}) \subseteq \psi(\mathbf{H}')$ if $\mathbf{H}\boldsymbol{R}\mathbf{H}'$ $(\mathbf{H}, \mathbf{H}' \in \mathbf{K})$.

(Intuitionistically, we require that $\psi(\mathbf{H})$ not only be non-empty, but that it contains at least one element; of course, a species may be known not to be empty without any particular element thereof being known.)

We define a *quantificational model* ϕ on a q. m. s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ to be a function $\phi(P^n, \mathbf{H})$, where P^n ranges over n-adic predicate letters (for all n), and $\mathbf{H}$ ranges over elements of $\mathbf{K}$. If $n = 0$, $\phi(P^n, \mathbf{H}) = \mathbf{T}$ or $\mathbf{F}$, and if $n \geq 1$, $\phi(P^n, \mathbf{H})$ is a subset of the Cartesian product $[\psi(\mathbf{H})]^n$. We again require for $n = 0$, that if $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, and $\phi(P^n, \mathbf{H}) = \mathbf{T}$, $\phi(P^n, \mathbf{H}') = \mathbf{T}$; for $n \geq 1$, analogously we require that if $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, $\phi(P^n, \mathbf{H}) \subseteq \phi(P^n, \mathbf{H}')$.

Let

$$\mathbf{U} = \bigcup_{\mathbf{H} \in \mathbf{K}} \psi(\mathbf{H}).$$

Given a quantificational model ϕ, we can define, for each formula A of intuitionistic quantification theory, a value $\phi(A, \mathbf{H}) = \mathbf{T}$ or $\mathbf{F}$, for each $\mathbf{H} \in \mathbf{K}$, *relative* to a fixed assignment of elements of $\mathbf{U}$ to the free individual variables of A. If A is an atomic formula, it is either a propositional letter P, in which case $\phi(P, \mathbf{H}) = \mathbf{T}$ or $\mathbf{F}$ is given, or it is a formula $P^n(x_1, \ldots, x_n)$

($n \geq 1$). In this latter case, let elements $a_1, \ldots, a_n$ of **U** be assigned to $x_1, \ldots, x_n$; then we can define, relative to this assignment, $\phi(P^n(x_1, \ldots, x_n), \mathbf{H}) = \mathbf{T}$ iff $(a_1, \ldots, a_n) \in \phi(P^n, \mathbf{H})$, and $\phi(P^n(x_1, \ldots, x_n), \mathbf{H}) = \mathbf{F}$ iff $(a_1, \ldots, a_n) \notin \phi(P^n, \mathbf{H})$. Given this assignment to atomic formulae, we can build up the assignment to more complex formulae by induction. Suppose $A(x_1, \ldots, x_n, y)$ is a formula, where at most the variables indicated occur free in $A(x_1, \ldots, x_n, y)$. Assume, that relative to each assignment of elements of **U** to $x_1, \ldots, x_n, y$, a truth-value $\phi(A(x_1, \ldots, x_n, y), \mathbf{H})$ has been defined for each **H**. We can then obtain values for $\phi((y)A(x_1, \ldots, x_n, y), \mathbf{H})$ and $\phi((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H})$ as follows. Let the elements $a_1, \ldots, a_n$ of **U** be assigned to the variables $x_1, \ldots, x_n$. Then:

e) We say $\phi((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ iff there is a $b \in \psi(\mathbf{H})$ such that $\phi(A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ when $x_1, \ldots, x_n$ are assigned $a_1, \ldots, a_n$, respectively, and y is assigned b; otherwise $\phi((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{F}$.

f) We say $\phi((y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ iff for each $\mathbf{H}' \in \mathbf{K}$ such that $\mathbf{H}R\mathbf{H}'$ $\phi(A(x_1, \ldots, x_n, y), \mathbf{H}') = \mathbf{T}$ when $x_1, \ldots, x_n$ are assigned $a_1, \ldots, a_n$, and y is assigned any element b of $\psi(\mathbf{H}')$; otherwise, $\phi((y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{F}$.

Finally, we stipulate that if truth-values $\phi(A, \mathbf{H})$ and $\phi(B, \mathbf{H})$ (for all $\mathbf{H} \in \mathbf{K}$), are given relative to an assignment to the free variables of A and B, then corresponding values $\phi(A \wedge B, \mathbf{H})$, $\phi(A \vee B, \mathbf{H})$, $\phi(A \supset B, \mathbf{H})$, and $\phi(\neg A, \mathbf{H})$ are to be defined according to the prescriptions (*a*) – (*d*).

To get a proper intuitionistic definition of model, we should again modify the given conditions and stipulate that a model ϕ is a function $\phi(P^n, \mathbf{H})$ as above, together with a function $\phi(A, \mathbf{H})$, assigning **T** or **F** to $\phi(A, \mathbf{H})$ relative to a given assignment of elements of **U** to the free variables of A, and satisfying the previously stated conditions (*e.g.*, that $\phi(P^n(x_1, \ldots x_n), \mathbf{H}) = \mathbf{T}$ when x_i is assigned $a_i (1 \leq i \leq n)$ iff $(a_1, \ldots, a_n) \in \phi(P^n, \mathbf{H})$). Again, this definition is classically substantially equivalent to the old one.

We note that all of the results to follow would remain valid if we allowed $\phi(P^n, \mathbf{H})$ ($n \geq 1$) to be any subset of $\mathbf{U}^n$, rather than restricting it to subsets of $[\psi(\mathbf{H})]^n$. We would also leave the theory unchanged if we regarded $\phi(A, \mathbf{H})$ as defined only when the free variables of A are assigned elements of $\psi(\mathbf{H})$, and undefined otherwise.

1.1. *Intuitive interpretation*

A triple $(\mathbf{G}, \mathbf{K}, S)$, with $\mathbf{K}$ a set, $\mathbf{G} \in \mathbf{K}$, and S a relation defined on $\mathbf{K}$, is called a *tree* (and $\mathbf{G}$ is called its *origin*) iff: (1) There is no $\mathbf{H} \in \mathbf{K}$ such that $\mathbf{H}S\mathbf{G}$; (2) for every $\mathbf{H} \in \mathbf{K}$ except $\mathbf{G}$, there is a unique $\mathbf{H}' \in \mathbf{K}$ such that $\mathbf{H}'S\mathbf{H}$; (3) for every $\mathbf{H} \in \mathbf{K}$, $\mathbf{G}S^*\mathbf{H}$, where S^* is the ancestral of the relation S (*i.e.*, $\mathbf{H}_1 S^* \mathbf{H}_2$ iff $\mathbf{H}_1 = \mathbf{H}_2$ or $\mathbf{H}_1 S^n \mathbf{H}_2$ for some power n of S). If $\mathbf{H}S\mathbf{H}'$, we call $\mathbf{H}$ the *predecessor of* $\mathbf{H}'$, and $\mathbf{H}'$ a *successor* of $\mathbf{H}$; the tree is *finitary* iff every $\mathbf{H}$ has only finitely many successors. An element $\mathbf{H}$ without successors is called an *endpoint*. Note that $\mathbf{K}$ can be characterized in terms of S as its field, and $\mathbf{G}$ can then be characterized as the unique element of $\mathbf{K}$ without a predecessor. (This definition of tree is adopted from [2]. Intuitionistically, we must further require that the elements $\mathbf{H}$ be natural numbers and that S be decidable.)

A m. s. $(\mathbf{G}, \mathbf{K}, R)$ is called a *tree* m. s. iff there exists a relation S such that $(\mathbf{G}, \mathbf{K}, S)$ is a tree and R is the smallest reflexive and transitive relation containing S (*i. e.*, $R = S^*$). In our remarks on intuitive interpretation, we will primarily be concerned with *tree* models (*i. e.*, models defined on a tree m. s. $(\mathbf{G}, \mathbf{K}, R)$). In fact, we will show below in section 1.2 that *any* model can be replaced by an "equivalent" tree model.

The rest of this section will consist of an informally stated intuitive interpretation of the modelling, together with indications how to state the interpretation more formally in terms of Kreisel's theory [11] of *absolutely* free choice sequences.[1]) The reader unfamiliar with [11] (or uninterested in these details is advised to omit the remarks relating to [11] but to read the rest of the section.

The interpretation proceeds as follows. Suppose we are given a model ϕ for a formula A of propositional calculus whose sole atomic subformulae are P, Q, R. For example, suppose we have a tree model ϕ on a m. s. $(\mathbf{G}, \mathbf{K}, R)$ diagrammed as follows:

[1]) We are informed that Gödel (unpublished) has proposed that such sequences be called "absolutely lawless," presumably on the ground that they are *not* completely free, being governed by the "higher order" requirement that no restrictions, other than those defining the spread in question, ever be placed on choices later by a free decision. Since Gödel's suggestion has not yet been adopted in print, we hesitate to make this change ourselves.

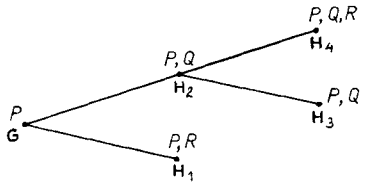

Figure 1.

The elements of K are $\mathbf{G}$, $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, $\mathbf{H}_4$. We have written an atomic formula above a node $\mathbf{G}$ or $\mathbf{H}_i$ if ϕ assigns it the value $\mathbf{T}$ on this node; we omit it if ϕ assigns it the value $\mathbf{F}$. Thus, *e.g.*, $\phi(P, \mathbf{G}) = \mathbf{T}$ while $\phi(Q, \mathbf{G}) = \phi(R, \mathbf{G}) = \mathbf{F}$.

We intend the nodes $\mathbf{H}$ to represent points in time (or "evidential situations"), at which we may have various pieces of information. If, at a particular point $\mathbf{H}$ in time, we have enough information to prove a proposition A, we say that $\phi(A, \mathbf{H}) = \mathbf{T}$; if we lack such information, we say that $\phi(A, \mathbf{H}) = \mathbf{F}$. If $\phi(A, \mathbf{H}) = \mathbf{T}$ we can say that A has been *verified* at the point $\mathbf{H}$ in time; if $\phi(A, \mathbf{H}) = \mathbf{F}$, then A has *not been verified* at $\mathbf{H}$. Notice, then, that $\mathbf{T}$ and $\mathbf{F}$ do not denote intuitionistic truth and falsity; if $\phi(A, \mathbf{H}) = \mathbf{T}$, then A has been verified to be true at the time $\mathbf{H}$; but $\phi(A, \mathbf{H}) = \mathbf{F}$ does not mean that A has been proved *false* at $\mathbf{H}$. It simply is not (yet) proved at $\mathbf{H}$, but may be established later.

Now given a point in time $\mathbf{G}$, there are various possibilities open for gaining further information about the propositions. One situation is diagrammed in Figure 1. At the point $\mathbf{G}$ (representing our present information), we have proved P. For all we know, we may remain "stuck" at $\mathbf{G}$ for an arbitrarily long time, without gaining any new information. But it is possible that we will gain enough information to "jump" to point $\mathbf{H}_1$ (in which case we have a proof of R in addition to P), or to the point $\mathbf{H}_2$ (where we get a proof of Q in addition to P), or even to the points $\mathbf{H}_3$ or $\mathbf{H}_4$. If we have "jumped" to the point $\mathbf{H}_2$, so that we have proved both P and Q, then as far as we know, we may remain "stuck"

for an arbitrarily long time at $\mathbf{H}_2$; but we may advance to $\mathbf{H}_3$ or $\mathbf{H}_4$. Notice that if we jump to the "situation" $\mathbf{H}_3$, we still have proved no more than P and Q; but this does not mean that the situation $\mathbf{H}_3$ is exactly like $\mathbf{H}_2$. In fact, as long as we remain at $\mathbf{H}_2$, the possibility is still open to us that we will some time or other be able to advance to $\mathbf{H}_4$ and prove R; but, if we are at the situation $\mathbf{H}_3$, we have gained enough information to exclude the option that R will ever be proved.

Now, in general, in a model structure $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, we interpret $\mathbf{G}$ as the present "evidential situation." If $\mathbf{H}$ is any situation, we say $\mathbf{H}\boldsymbol{R}\mathbf{H}'$ if, as far as we know, at the time $\mathbf{H}$, we may later get enough information to advance to $\mathbf{H}'$. Thus, since the information we have at $\mathbf{H}$ may be all the knowledge we have for an arbitrarily long time, we stipulate that $\mathbf{H}\boldsymbol{R}\mathbf{H}$; and the transitivity property of $\boldsymbol{R}$ is intuitively obvious. The requirement that, for any A, if $\phi(A, \mathbf{H}) = \mathbf{T}$ and $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, then $\phi(A, \mathbf{H}') = \mathbf{T}$, simply means that if we already have a proof of A in the situation $\mathbf{H}$, then we can accept A as proved in any later situation $\mathbf{H}'$—we don't forget. Finally, the inductive clauses for propositional calculus are in consonance with the intuitionistic interpretations of these notions. Thus $A \wedge B$ [$A \vee B$] is proved when both A and B have been proved [either A has been proved or B has been proved]; so $\phi(A \wedge B, \mathbf{H}) = \mathbf{T}$ iff $\phi(A, \mathbf{H}) = \phi(B, \mathbf{H}) = \mathbf{T}$ [$\phi(A \vee B, \mathbf{H}) = \mathbf{T}$ iff $\phi(A, \mathbf{H}) = \mathbf{T}$ or $\phi(B, \mathbf{H}) = \mathbf{T}$]. Notice that disjunction and conjunction behave, in a given situation $\mathbf{H}$, as if they were classical truth-functions. Negation and implication, on the other hand, are not so treated. To assert $\neg A$ intuitionistically in the situation $\mathbf{H}$, we need to know at $\mathbf{H}$ not only that A has not been verified at $\mathbf{H}$, but that it cannot possibly be verified at any later time, no matter how much more information is gained; so we say that $\phi(\neg A, \mathbf{H}) = \mathbf{T}$ iff $\phi(A, \mathbf{H}') = \mathbf{F}$ for every $\mathbf{H}' \in \mathbf{K}$ s.t. $\mathbf{H}\boldsymbol{R}\mathbf{H}'$. Again, to assert $A \supset B$ in a situation $\mathbf{H}$, we need to know that in any later situation $\mathbf{H}'$ where we get a proof of A, we also get a proof of B; the inductive definition of $\phi(A \supset B, \mathbf{H})$ formalizes this requirement.

Consider the following two point countermodel to the law of excluded middle:

Figure 2.

We have $\phi(P, \mathbf{H}) = \mathbf{T}$, $\phi(P, \mathbf{G}) = \mathbf{F}$. Since $\phi(P, \mathbf{H}) = \mathbf{T}$, $\phi(\neg P, \mathbf{G}) = \mathbf{F}$, and hence $\phi(P \vee \neg P, \mathbf{G}) = \mathbf{F}$. Intuitively, at the present situation **G**, we have not yet proved P; nor can we assert $\neg P$, since the possibility remains that we will get enough information later to advance to **H** and assert P. Thus, at the point **G**, we are not in a position to assert $P \vee \neg P$.

These considerations can readily be formulated in terms of Kreisel's theory FC of absolutely free choice sequences. Intuitively, an absolutely free choice sequence (a.f.c.s.) is a free choice sequence α, chosen from a given spread S, in which it is stipulated from the beginning that no restrictions, other than the conditions defining the spread S, can ever be placed on the choices.

Figure 2, then, for example, can be interpreted in terms of the present theory as follows: Consider a.f.c.s.'s from the spread S consisting of free choices of 0's and 1's, in which, however, 1 can be followed only by 1. Intuitively, we interpret the situation **G** as a choice of 0 and **H** as a choice of 1. Since, starting with **G**, we can remain "stuck" at **G** as long as we like, we permit 0 to be followed by an arbitrary number of 0's as well as by 1; but, since **H** is followed only by itself, we permit 1 to be followed only by 1. Then $P(\alpha)$ is the assertion "a 1 occurs on the a.f.c.s. α" (*i.e.*, $(\exists n)(\alpha(n) = 1)$, where n ranges over natural numbers). As long as we have chosen only 0's in α, we have not established $P(\alpha)$; but on the other hand, since α is chosen with no restrictions other than being in S, we cannot exclude the possibility of the choice of a 1 later, so we cannot establish $\neg P(\alpha)$. These considerations can be formalized easily in Kreisel's FC so as to yield a proof of $\neg(\alpha \upharpoonright S)\ (P(\alpha) \vee \neg P(\alpha))$, where $\alpha \upharpoonright S$ ranges over a.f.c.s.'s in S.

More generally, given any (intuitionistically defined) countable tree model ϕ of A on a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, suppose we identify the nodes (elements of **K**) with natural numbers, identifying **G** in particular with 0. Define in terms of $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ a spread S consisting of all free choice sequences in which the initial choice is 0, and the choice of any natural number m must be followed either by a further choice of m or by a choice of some successor of m on the tree. To any atomic subformula P of A, and a.f.c.s. α in S, associate a formula $P(\alpha)$ abbreviating $(\exists x)(\exists m)(\alpha(x) = m$ and $\phi(P, m) = \mathbf{T})$. Given B, C, and associated formulae $B(\alpha)$ and $C(\alpha)$, associate with $B \wedge C$, $B(\alpha) \wedge C(\alpha)$; with $B \vee C$, $B(\alpha) \vee C(\alpha)$, etc. Then, it is easily seen by induction that, for any subformula B of A,

if $\phi(B, m) = \mathbf{T}$, then $(\alpha \upharpoonright S)((\exists x)(\alpha(x) = m) \supset B(\alpha))$, and if $\phi(B, m) = \mathbf{F}$, $\neg(\alpha \upharpoonright S)((\exists x)(\alpha(x) = m) \supset B(\alpha))$. In particular, if $\phi(B, \mathbf{G}) = \mathbf{T}[=\mathbf{F}]$, then since every a.f.c.s. in S contains $0 (= \mathbf{G})$, we have $(\alpha \upharpoonright S)\, B(\alpha)$ $[\neg(\alpha \upharpoonright S)\, B(\alpha)]$. If the m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ and model ϕ can be formally described in Kreisel's FC, the preceding reasoning can be formalized in FC, and thus in particular, if $\phi(A, \mathbf{G}) = \mathbf{F}$, $\vdash \neg(\alpha \upharpoonright S)A(\alpha)$ in FC, giving a counter-example to the validity of A.

To extend this treatment to quantifiers, consider first the following countermodel to $(x)\,(P(x) \vee Q).\supset.(x)P(x) \vee Q$:

Figure 3.

We have $\phi(P(x), \mathbf{G}) = \phi(P(x), \mathbf{H}) = \mathbf{T}$, when x is assigned a, but $\phi(P(x), \mathbf{G}) = \phi(P(x), \mathbf{H}) = \mathbf{F}$ when x is assigned b. Further, $\phi(Q, \mathbf{G}) = \mathbf{F}$, $\phi(Q, \mathbf{H}) = \mathbf{T}$, $\mathbf{G}\boldsymbol{R}\mathbf{H}$ but not $\mathbf{H}\boldsymbol{R}\mathbf{G}$, and $\psi(\mathbf{G}) = \{a\}$, $\psi(\mathbf{H}) = \{a, b\}$. All this information is included in the diagram. It is easily verified that $\phi((x)\,(P(x) \vee Q), \mathbf{G}) = \mathbf{T}$, but $\phi((x)P(x) \vee Q, \mathbf{G}) = \mathbf{F}$. Intuitively, we can interpret the situation as follows: Identify the elements a and b with the integers 0 and 1, respectively. Let R be Fermat's last theorem, and let Q be $R \vee \neg R$. Let V be the species containing 0, and containing 1 if Q is true (*i.e.*, $V = \{ m \mid m = 0 \vee (m = 1 \wedge Q)\}$), and let x be a variable ranging over V. Let $P(x)$ be the statement $x = 0$. Then, already at the present situation $\mathbf{G}$, we can assert $V \subseteq \{0, 1\}$, and $1 \in V$ iff Q; so we can assert $(x)\,(P(x) \vee Q)$. But so long as we have not advanced to the situation $\mathbf{H}$, where Fermat's last theorem has been decided, so that we can assert Q, we cannot assert $(x)P(x) \vee Q$.

N.B. It should be remarked that $(x)\,(P(x) \vee Q).\supset.(x)P(x) \vee Q$ holds in any quantificational model such that $\psi(\mathbf{H})$ is constant.

Thus, in general, if the variables in a formula A range over a domain $\mathbf{D}$, then for each situation $\mathbf{H}$, $\psi(\mathbf{H})$ is the species of all individuals known to be in $\mathbf{D}$ on the basis of the information available at $\mathbf{H}$. (So, in the case of the paragraph above, at the present situation $\mathbf{G}$, $\psi(\mathbf{G}) = \{0\}$; but when at $\mathbf{H}$, Q has been proved, $\psi(\mathbf{H}) = \{0, 1\}$. Since $\mathbf{D}$ is to contain an

element, we must know at least one element of **D** from the outset, so that $\psi(\mathbf{G})$ must contain at least one element. The restriction that $\mathbf{H}R\mathbf{H}'$ is to imply $\psi(\mathbf{H}) \subseteq \psi(\mathbf{H}')$ should now be obvious on the intended interpretation. Notice that, to assert in a situation **H** that for every element x of **D**, $P(x)$ is true, we must know not only that $P(x)$ is true for every x in $\psi(\mathbf{H})$, but also that it is true for every x which may later be proved to be in **D**; *i.e.*, for every x in $\psi(\mathbf{H}')$, where $\mathbf{H}R\mathbf{H}'$; and this is exactly the inductive clause for universal quantification. On the other hand, to assert the existence of an x in **D** such that $P(x)$ is true, we need to find an element x which has already been proved to be in **D** (*i.e.*, which is in $\psi(\mathbf{H})$), and such that $P(x)$ is true; and this is exactly what the condition on existential quantification requires.

These facts can again be stated more formally in terms of the theory of absolutely free choice sequences. Suppose we are given an (intuitionistically defined) countable tree m. s. (**G**, **K**, R) in which **U**, and hence $\psi(\mathbf{H})$ for each **H** is countable. Then, we can identify both the elements of **K** and the elements of **U** with natural numbers, identifying **G** in particular with 0. We then associate with (**G**, **K**, R) a spread S of absolutely free choice sequences, defined just as above. Further, for any a.f.c.s. α in S, let $\mathbf{D}_\alpha$ be the species of all natural numbers n such that there is a natural number x such that $n \in \psi(\alpha(x))$. Let x_α be a variable ranging over $\mathbf{D}_\alpha$ (*i.e.*, $(x_\alpha)(\ldots)$ is to be interpreted as $(x)(x \in \mathbf{D}_\alpha \supset \ldots)$ and similarly for $(\exists x_\alpha)$). Then since $\alpha(0) = 0 = \mathbf{G}$, and $\psi(\mathbf{G})$ contains a natural number, $\mathbf{D}_\alpha$ has an element for all α. Let ϕ be an (intuitionistically defined) q. model on (**G**, **K**, R)for some formula A. Given any atomic subformula $P^n(x_1, \ldots, x_n)$, and an a.f.c.s. α of S, we associate with these two an assertion $P(\alpha, x_{1_\alpha}, \ldots, x_{n_\alpha})$, where the variables $x_{1_\alpha}, \ldots, x_{n_\alpha}$ range over $\mathbf{D}_\alpha$, and where $P(\alpha, x_{1_\alpha}, \ldots, x_{n_\alpha})$ says that $\phi(P^n(x_1, \ldots, x_n), m) = \mathbf{T}$ for some m on α, when x_{i_α} is assigned to the variable x_i ($i = 1, \ldots, n$; note that $x_{i_\alpha} \in \mathbf{D}_\alpha \subseteq \mathbf{U}$). Given formulae $A(\alpha, x_{1_\alpha}, \ldots, x_{n_\alpha})$ and $B(\alpha, y_{1_\alpha}, \ldots, y_{m_\alpha})$ associated with $A(x_1, \ldots, x_n)$ and $B(y_1, \ldots, y_m)$, respectively, associate $A(\alpha, x_{1_\alpha}, \ldots, x_{n_\alpha}) \wedge B(\alpha, y_{1_\alpha}, \ldots, y_{m_\alpha})$ with $A(x_1, \ldots, x_n) \wedge B(y_1, \ldots, y_m)$, and similarly for the other connectives.

Further, associate $(x_{i_\alpha})A(\alpha, x_{1_\alpha}, \ldots, x_{n_\alpha})$ with $(x_i)A(x_1, \ldots, x_n)$, and similarly for the existential quantifier. Then, we prove, by induction, that, for any $m \in \mathbf{K}$, if $A(x_1, \ldots, x_n)$ contains only the free variables listed and $x_1, \ldots, x_n$ are assigned $a_1, \ldots, a_n \in \psi(m)$, then if $\phi(A(x_1, \ldots,$

$x_n), m) = \mathbf{T}[=\mathbf{F}]$ relative to this assignment, we have in FC $(\alpha \upharpoonright S)$ $((\exists x)\ (\alpha(x) = m) \supset A(\alpha, a_1, \ldots, a_n))\ [\neg(\alpha \upharpoonright S)((\exists x)\ (\alpha(x) = m) \supset A(\alpha, a_1, \ldots, a_n))]$. In particular, if $m = 0 = \mathbf{G}$, since $(\alpha \upharpoonright S)(\exists x)\ (\alpha(x) = 0)$, we get $(\alpha \upharpoonright S)A(\alpha, a_1, \ldots, a_n)\ [\neg(\alpha \upharpoonright S)A(\alpha, a_1, \ldots, a_n)]$. Thus, if A does not contain free variables, and $\phi(A, \mathbf{G}) = \mathbf{F}$, we get a proof in FC that A is not generally valid.

To translate, then, the example given above into FC, notice that, where B is the full binary spread,

(a) $$(\alpha \upharpoonright B)(x)((\exists y)\ (\alpha(y) = x) \supset (x = 0 \vee (\exists y)\ (\alpha(y) = 1))),$$

but also

(b) $$\neg(\alpha \upharpoonright B)((x)((\exists y)\ (\alpha(y) = x) \supset x = 0) \vee (\exists y)\ (\alpha(y) = 1)).$$

Thus we have refuted the "law" $(x)\ (P(x) \vee Q).\supset.(x)P(x) \vee Q$; for if it held, it would hold for any free choice sequence α, with x ranging over the species of all z such that $(\exists y)\ (\alpha(y) = z)$, contrary to (a) and (b). Notice that, since (a) is a triviality and (b) follows from the fan theorem, we could simply have used the *ordinary* theory of free choice sequences instead of FC.

We remark that, following Dyson and Kreisel [9], the countermodels in FC that we have described, assigning certain infinite sequences of natural numbers to formulae, can classically be interpreted as countermodels in Baire space (the space of all sequences of natural numbers, with the usual topology). In fact, by examination of the countermodels actually produced below, it follows that every unprovable formula has a countermodel in the Cantor set, as Dyson and Kreisel assert.

REMARK. The following remarks on the uses of absolutely free choice sequences are not relevant to the main point of the present paper, but will be added here:

1. All the theorems which are proved in the last chapter of Heyting [12], using Brouwer's method of free choice sequences depending on the solving of problems, can be carried out in FC. To take the first example given by Heyting: to show that it is absurd that, for every real number a, $a \neq 0$ should imply $a \# 0$. For if this were true, then for any free choice sequence α in the binary spread, by associating with α the real number

$$\sum_{x=0}^{\infty} \alpha(x)/2^x,$$

we could show that $\neg(x)(\alpha(x) = 0) \supset (\exists x)\,(\alpha(x) = 1)$; hence, in particular, this would hold for *absolutely* free choice sequences. But it is easy to show, in FC, that $(\alpha \restriction B)\,\neg(x)(\alpha(x) = 0)$. Hence we need only show in FC that $\neg(\alpha \restriction B)(\exists x)(\alpha(x) = 1)$; but this easily follows from the fan theorem, since $(\alpha \restriction B)(\exists x)(\alpha(x) = 1)$ would imply $(\exists m)(\alpha \restriction B)(\exists x \leq m)$ $(\alpha(x) = 1)$, which is absurd. Similar treatments are possible for all the refutations of classical theorems treated by Heyting by this method in [12].

I think it probable that such treatments in FC will extend to *all* the counterexamples to classical theorems which Brouwer gives by his method; but I have not made a survey of the literature.

A careful reader of the present section on the interpretation of our models will find it plausible that, conversely, a good deal of the interpretation, at least for propositional calculus, that has just been carried out in FC, could be carried out using Brouwer's method of ips depending on the solving of problems.

2. The following example, which refutes both Kuroda's conjecture (cf. [13]) and Markov's principle (cf. [14]) in FC, was inspired by applying the methods of the present section to obtain a countermodel to $(x)\,\neg\neg A(x) \supset \neg\neg(x)A(x)$. Let S be the finitary spread consisting of all free choice sequences α such that $\alpha(x + 1) = \alpha(x)$ or $\alpha(x + 1) = \alpha(x) + 1$ for every x. We show in FC

(a) $$(\alpha \restriction S)(m)\neg\neg(\exists n)\,(\alpha(n) \geq m)$$

(b) $$(\alpha \restriction S)\neg(m)\,(\exists n)\,(\alpha(n) \geq m).$$

To prove (a), let α be an a.f.c.s. in S, let m be an integer and suppose for *reductio ad absurdum* that $\neg(\exists n)(\alpha(n) \geq m)$. Then, since α is absolutely free, by axiom 5.1 of FC, there is an initial segment $\bar{\alpha}(x)$ of α such that (*) $(\beta \restriction S)\,(\bar{\beta}(x) = \bar{\alpha}(x) \supset \neg(\exists n)\,(\bar{\beta}(n) \geq m))$. Now $\alpha(x) < m$, for otherwise $(\exists n)\,(\alpha(n) \geq m)$. Hence, since every ips on S is non-decreasing, for all $y < x$, $\alpha(y) < m$. Now (*) asserts that, if we have chosen the first x components of β so that $\bar{\beta}(x) = \bar{\alpha}(x)$, we can never choose $\bar{\beta}(n) \geq m$ for any n. But by axiom 5.3 of FC, there are a.f.c.s.'s β in S, satisfying the conditions $\bar{\beta}(x) = \bar{\alpha}(x)$ and $\beta(x + i) = \alpha(x) + i\,(0 \leq i \leq m - \alpha(x))$,

since this finite sequence of choices accords with the spread law of S. But then if $n = x + m - \alpha(x)$, $\beta(n) = m$, contrary to (*).

To prove (b), let α be an a.f.c.s. in S, and for *reductio ad absurdum* assume $(m)\,(\exists n)\,(\alpha(n) \geq m)$. Then, again by axiom 5.1 of FC, there is and x such that (**) $(\beta \restriction S)\,(\bar{\beta}(x) = \bar{\alpha}(x) \supset (m)\,(\exists n)(\bar{\beta}(n) \geq m))$. Given any a.f.c.s. β in S, assign a value $f(\beta)$ as follows: If $\bar{\beta}(x) \neq \bar{\alpha}(x)$, let $f(\beta) = 0$; if $\bar{\beta}(x) = \bar{\alpha}(x)$, let $f(\beta)$ be the least n such that $\beta(n) \geq \alpha(x) + 1$. By (**), f is well defined for all such β, so by the fan theorem there is some finite integer p such that $f(\beta)$ is wholly determined by $\bar{\beta}(p)$. We can thus write $f(\beta)$ as $f(\bar{\beta}(p))$. Clearly, by the definition of f, $p \geq x$. Now, again using axiom 5.3 of FC, determine β by requiring $\bar{\beta}(x) = \bar{\alpha}(x)$, $\beta(x + i) = \alpha(x)$ $(0 \leq i \leq p - x)$. Then (**) asserts that $\beta(f(\bar{\beta}(p))) \geq \alpha(x) + 1$. But this is clearly absurd, since again by 5.3 we are perfectly free to continue the choices by $\beta(p + j) = \alpha(x)$ $(0 \leq j \leq f(\bar{\beta}(p)) - p)$, so that, taking $j = f(\bar{\beta}(p)) - p$, we would get $\beta(f(\bar{\beta}(p))) = \alpha(x) < \alpha(x) + 1$. So (b) is proved.

We will now use (a) and (b) to refute Kuroda's conjecture [13] and Markov's principle [14]. Kuroda's conjecture asserts that for m a number variable, $(m) \neg\neg A(m)$ implies $\neg\neg (m)A(m)$. Using Kuroda's conjecture, we could derive from (a) the assertion $(\alpha \restriction S) \neg\neg (m)(\exists n)(\alpha(n) \geq m)$, which directly contradicts (b); so Kuroda's conjecture is refutable in FC. Similarly Markov's principle asserts that, for a decidable predicate $A(x)$ and number variable n, $\neg\neg (\exists n)A(n)$ implies $(\exists n)A(n)$. But, if we take $A(n)$ to be $\alpha(n) \geq m$, then $A(n)$ is primitive recursive and hence decidable. Then Markov's principle would allow us to derive $(\alpha \restriction S)\,(m)\,(\exists n)\,(\alpha(n) \geq m)$ from (a), again contradicting (b).

In spite of the proofs by Gödel and Kreisel that strong completeness of Heyting's predicate calculus implies certain forms of Markov's principle, I am unable to see how to convert these results into a proof in FC that Heyting's predicate calculus is not strongly complete, and I doubt that such a conversion is in fact possible. If S' is the spread consisting of all α such that there is a β in S such that $(x)\,(\alpha(x + 1) = \beta(x))$, it is easy to conclude from the present results that $(\alpha \restriction S')\,\neg(\exists n)(\alpha(n) \geq \alpha(0))$ and that $\neg(\alpha \restriction S')\,(\exists n)(\alpha(n) \geq \alpha(0))$; but, since α here ranges over *absolutely* free choice sequences of S' and not ordinary free choice sequences, we are unable to apply Theorem 1 of Kreisel [15] to conclude that Heyting's predicate calculus is not strongly complete.

1.2. *Relationship to the Beth models*

In this section, we discuss the relationship of the present model theory to that of Beth [8]. We will show that the present models can be "translated," in a natural way, into Beth models. Using an intuitive interpretation of the Beth modelling, we will also show that the mapping leads to an interpretation of our own quantificational models which is alternative to that of the previous section; in this interpretation, the variables always range over the species of natural numbers.

This section can be omitted, if desired, without loss of continuity.

First, we present the notion of Beth model in our terminology as follows: Let $(\mathbf{G}, \mathbf{K}, \boldsymbol{S})$ be a tree, and let $\boldsymbol{R} = \boldsymbol{S}^*$, so that $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is a tree m.s. By a *path* in the tree $(\mathbf{G}, \mathbf{K}, \boldsymbol{S})$ we mean a sequence $\{\mathbf{H}_i\}$ of elements of $\mathbf{K}$, indexed on either the sequence of natural numbers or on some finite initial segment thereof, satisfying the conditions: (*a*) $\mathbf{H}_0 = \mathbf{G}$; (*b*) for $i > 0$, $\mathbf{H}_{i-1}\boldsymbol{S}\mathbf{H}_i$; (c) if $\{\mathbf{H}_i\}$ has a last element $\mathbf{H}_n$, $\mathbf{H}_n$ is an endpoint of $(\mathbf{G}, \mathbf{K}, \boldsymbol{S})$. If some $\mathbf{H}_i = \mathbf{H}$, we say the path is *through* $\mathbf{H}$. Let $\mathbf{B}$ be a subset of $\mathbf{K}$. If every path through $\mathbf{H}$ intersects $\mathbf{B}$, we say that $\mathbf{H}$ is *barred* by $\mathbf{B}$. Thus, for example, $\mathbf{H}$ is barred by $\{\mathbf{H}\}$.

By a *Beth model* on $(\mathbf{G}, \mathbf{K}, \boldsymbol{S})$, we mean a binary function $\eta(P, \mathbf{H})$ satisfying the following conditions: (*a*) $\eta(P, \mathbf{H}) = \mathbf{T}$ or $\mathbf{F}$, where P is atomic and $\mathbf{H} \in \mathbf{K}$. (*b*) If $\mathbf{H}\boldsymbol{S}\mathbf{H}'$ and $\eta(P, \mathbf{H}) = \mathbf{T}$, then $\eta(P, \mathbf{H}') = \mathbf{T}$. (*c*) If $\mathbf{H}$ is barred by $\mathbf{B}$ and $\eta(P, \mathbf{H}') = \mathbf{T}$ for every $\mathbf{H}' \in \mathbf{B}$, then $\eta(P, \mathbf{H}) = \mathbf{T}$.

Given a Beth model η, we define by induction values $\eta(A, \mathbf{H})$ for an arbitrary formula A of the propositional calculus. Suppose $\eta(A, \mathbf{H})$ and $\eta(B, \mathbf{H})$ have already been defined. Define $\eta(\neg A, \mathbf{H})$, $\eta(A \wedge B, \mathbf{H})$, and $\eta(A \supset B, \mathbf{H})$ exactly as was done above for a model ϕ; simply replace "ϕ" by "η" throughout. Finally define $\eta(A \vee B, \mathbf{H}) = \mathbf{T}$ iff there is a subset $\mathbf{B}$ of $\mathbf{K}$ such that $\mathbf{H}$ is barred by $\mathbf{B}$ and $\eta(A, \mathbf{H}') = \mathbf{T}$ or $\eta(B, \mathbf{H}') = \mathbf{T}$ for every $\mathbf{H}' \in \mathbf{B}$; otherwise, $\eta(A \vee B, \mathbf{H}) = \mathbf{F}$. Notice that if $\eta(A, \mathbf{H}) = \mathbf{T}$ or $\eta(B, \mathbf{H}) = \mathbf{T}$, $\eta(A \vee B, \mathbf{H}) = \mathbf{T}$; we can take $\{\mathbf{H}\}$ as the set $\mathbf{B}$ barring $\mathbf{H}$.

Notice that condition (*b*) above actually implies the strengthened condition (*b'*): If $\eta(P, \mathbf{H}) = \mathbf{T}$ and $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, then $\eta(P, \mathbf{H}') = \mathbf{T}$. Using this fact, it is easy to prove by induction that the properties (*a*) – (*c*) actually hold not only for an atomic formula P, but also for an arbitrary formula A.

As in the case of models ϕ, the inductive definition of $\eta(A, \mathbf{H})$ just

given depends on the law of the excluded middle. Again as in the case of a model, we can correct the situation by modifying the definition of a Beth model. We leave the modification to the reader.

A Beth model η on a tree $(\mathbf{G}, \mathbf{K}, \mathcal{S})$ is called *finitary* if $(\mathbf{G}, \mathbf{K}, \mathcal{S})$ is *finitary*. Beth's own version of his models in [8] is actually equivalent to our notion of a finitary Beth model.

We call a Beth model η a *strong* Beth model iff for all $\mathbf{H} \in \mathbf{K}$ and formulae A and B, $\eta(A \vee B, \mathbf{H}) = \mathbf{T}$ implies $\eta(A, \mathbf{H}) = \mathbf{T}$ or $\eta(B, \mathbf{H}) = \mathbf{T}$. Notice that, on account of the validity of condition (b') above, a Beth model η is also a model in our sense. However, since the inductive clause for disjunction in a Beth model differs from the inductive clause for our sense of model non-atomic formulae may be given different values according as η is considered as a model in our sense or as a Beth model. A strong Beth model is precisely a Beth model in which this eventuality never happens.

The intuitive rationale behind the Beth models is simple: Again the elements ("nodes") of the tree model $(\mathbf{G}, \mathbf{K}, \mathcal{S})$ are points in time, or evidential situations; but we no longer suppose that we are allowed to remain at a given point $\mathbf{H}$ as long as we please. On the contrary, if $\mathbf{H}$ is a node of the tree $(\mathbf{G}, \mathbf{K}, \mathcal{S})$, we are forced, unless $\mathbf{H}$ is an endpoint, to "jump" within a fixed, finite time to one of the successors of $\mathbf{H}$ in the tree. (Paradigmatic of such a game, of course, are free choices in an (absolutely) free choice sequence α: after each choice we are forced to make another, within a finite length of time, unless the spread-law states that the choice we have just made is terminal.) $\eta(P, \mathbf{H}) = \mathbf{T}[=\mathbf{F}]$ means that P has been established [has not yet been established] at the time $\mathbf{H}$, so the conditions (a) and (b) on η are clear. If $\mathbf{H}$ is barred by $\mathbf{B} \subseteq \mathbf{K}$, condition (c) says if we know that P will be established at any $\mathbf{H}' \in \mathbf{B}$, then we already know at $\mathbf{H}$ that P is true; for, once we are at $\mathbf{H}$, we must get to some $\mathbf{H}' \in \mathbf{B}$ in a finite time. Similarly, the inductive clause which defines $\eta(A \vee B, \mathbf{H})$ observes that to establish $A \vee B$ at $\mathbf{H}$ it is sufficient to know that, in a finite number of "moves," we must either establish A or establish B; that is to say, it suffices to know that there is a $\mathbf{B}$ which bars $\mathbf{H}$ such that every $\mathbf{H}' \in \mathbf{B}$ either establishes A or establishes B. The inductive clauses for the other connectives are as before.

As in section 1.1, we can give a more precise justification of the

definition in terms of absolutely free choice sequences. As before, we identify the elements of the (countable) tree $(\mathbf{G}, \mathbf{K}, S)$ with natural numbers, associating 0 with $\mathbf{G}$. We then consider the spread S of all absolutely free choice sequences of elements of $\mathbf{K}$ whose first term is 0 and which satisfy the condition that $\alpha(n) S \alpha(n+1)$, unless $\alpha(n)$ is an endpoint of $(\mathbf{G}, \mathbf{K}, S)$, in which case $\alpha(n) = \alpha(n+1)$. For any atomic P, associate a formula $P(\alpha)$ which says $(\exists n)\,(\exists x)\,(\alpha(x) = n \wedge \eta(P, n) = \mathbf{T})$. We then define inductively a formula $A(\alpha)$ associated with an arbitrary formula A, exactly as in section 1.1. Again as in 1.1, if $\eta(A, n) = \mathbf{T}$ $[=\mathbf{F}]$ we can derive $(\alpha \upharpoonright S)\,((\exists x)(\alpha(x) = n) \supset A(\alpha))$ $[\neg(\alpha \upharpoonright S)((\exists x)(\alpha(x) = n) \supset A(\alpha))]$ in FC.

We now show how the ideas of section 1.1 can be modified so as to show how every model can be transformed into an "equivalent" strong Beth model. Let ϕ be a model on a m.s. $(\mathbf{G}, \mathbf{K}, R)$. Define a tree $(\mathbf{G}', \mathbf{K}', S')$ as follows: Let $\mathbf{K}'$ be the set of all finite non-empty sequences $\{\mathbf{H}_i\}_{i=1}^{n}$, where $\mathbf{H}_i \in \mathbf{K} (1 \leq i \leq n)$, $\mathbf{H}_1 = \mathbf{G}$, and $\mathbf{H}_i R \mathbf{H}_{i+1} (1 \leq i < n)$. Let $\mathbf{G}'$ be the sequence whose sole term is $\mathbf{G}$. We say, for $\mathbf{H}_1', \mathbf{H}_2' \in \mathbf{K}'$, that $\mathbf{H}_1' S' \mathbf{H}_2'$ iff $\mathbf{H}_1'$ is the initial segment of $\mathbf{H}_2'$ formed by omitting the last term of $\mathbf{H}_2'$. (Then, if $R' = S'^{*}$, $\mathbf{H}_1' R' \mathbf{H}_2'$ iff $\mathbf{H}_1'$ is an initial segment of $\mathbf{H}_2'$.) For any $\mathbf{H}' \in \mathbf{K}'$, let $\ell(\mathbf{H}')$ be the last term of $\mathbf{H}'$, then $\ell(\mathbf{H}') \in \mathbf{K}$. Define $\eta(P, \mathbf{H}')$ (P atomic, $\mathbf{H}' \in \mathbf{K}'$) by $\eta(P, \mathbf{H}') = \phi(P, \ell(\mathbf{H}'))$.

Let $\mathbf{H}' \in \mathbf{K}'$, $\mathbf{H}' = \{\mathbf{H}_i\}_{i=1}^{n}$. We define an associated path $\mathbf{P}(\mathbf{H}') = \{\mathbf{H}_j'\}_{j=0}^{\infty}$ as follows: For $0 \leq j < n$, let $\mathbf{H}_j$ be the unique initial segment of $\mathbf{H}'$ with $j+1$ terms. For $j \geq n$, let $\mathbf{H}_j'$ be the $j+1$-termed sequence whose first n terms are $\mathbf{H}_1, \ldots, \mathbf{H}_n$ and whose other terms are all equal to $\mathbf{H}_n$. So for $j \geq n$, let $\ell(\mathbf{H}_j') = \mathbf{H}_n$. Clearly $\mathbf{P}(\mathbf{H}')$ is a path through $\mathbf{H}'$; further, for any $\mathbf{H}_j'$ on this path, $\ell(\mathbf{H}_j') R \ell(\mathbf{H}')$.

We now assert:

THEOREM 1 (*First part*): *η is a strong Beth model. Further η is equivalent to ϕ in the sense that $\eta(A, \mathbf{H}') = \phi(A, \ell(\mathbf{H}'))$ for any $\mathbf{H}' \in \mathbf{K}'$ and formula A. In particular $\eta(A, \mathbf{G}') = \phi(A, \mathbf{G})$ for any A.*

PROOF. First we show that η is a Beth model. Condition (a) is clear. For (b), if $\eta(P, \mathbf{H}_1') = \mathbf{T}$ and $\mathbf{H}_1' S' \mathbf{H}_2'$, then $\ell(\mathbf{H}_1') R \ell(\mathbf{H}_2')$. Since $\phi(P, \ell(\mathbf{H}_1')) = \eta(P, \mathbf{H}_1') = \mathbf{T}$, and since ϕ is a model, $\phi(P, \ell(\mathbf{H}_2') = \mathbf{T}$, hence $\eta(P, \mathbf{H}_2') = \mathbf{T}$. For (c) let $\mathbf{H}'$ be barred by $\mathbf{B} \subseteq \mathbf{K}'$. Then the path $\mathbf{P}(\mathbf{H}')$ intersects $\mathbf{B}$. Let $\mathbf{H}''$ be some point of the intersection. To establish (c)

it is sufficient to show that if $\eta(P, \mathbf{H}'') = \mathbf{T}$, $\eta(P, \mathbf{H}') = \mathbf{T}$. Since $\mathbf{H}''$ is on the path $\mathbf{P}(\mathbf{H}')$, $\ell(\mathbf{H}'')R\ell(\mathbf{H}')$. Since $\eta(P, \mathbf{H}'') = \mathbf{T}$, $\phi(P, \ell(\mathbf{H}'')) = \mathbf{T}$. Hence since $\ell(\mathbf{H}'')R\ell(\mathbf{H}')$, $\eta(P, \mathbf{H}') = \phi(P, \ell(\mathbf{H}')) = \mathbf{T}$, which proves (c). So η is a Beth model. We now establish $\eta(A, \mathbf{H}') = \phi(A, \ell(\mathbf{H}'))$ for any A; if we establish this, it will obviously follow that the Beth model η is strong. The result holds by definition for atomic A; for other A we prove it by induction. The only non-trivial case is that of disjunction. Suppose $\eta(A, \mathbf{H}') = \phi(A, \ell(\mathbf{H}'))$ and $\eta(B, \mathbf{H}') = \phi(B, \ell(\mathbf{H}'))$. If $\phi(A \vee B, \ell(\mathbf{H}')) = \mathbf{T}$, then either $\phi(A, \ell(\mathbf{H}')) = \eta(A, \mathbf{H}') = \mathbf{T}$, or $\phi(B, \ell(\mathbf{H}')) = \eta(B, \mathbf{H}') = \mathbf{T}$. In either case, $\eta(A \vee B, \mathbf{H}') = \mathbf{T}$. Conversely, if $\eta(A \vee B, \mathbf{H}') = \mathbf{T}$, then there is a set $\mathbf{B} \subseteq \mathbf{K}'$, barring $\mathbf{H}'$ and such that $\eta(A, \mathbf{H}'') = \mathbf{T}$ or $\eta(B, \mathbf{H}'') = \mathbf{T}$ for every $\mathbf{H}'' \in \mathbf{B}$. Choose $\mathbf{H}''$ in the intersection of B with $\mathbf{P}(\mathbf{H}')$. Suppose, say, that $\eta(A, \mathbf{H}'') = \mathbf{T}$. Then $\phi(A, \ell(\mathbf{H}'')) = \mathbf{T}$. Since $\ell(\mathbf{H}'')R\ell(\mathbf{H}')$, $\phi(A, \ell(\mathbf{H}')) = \mathbf{T}$, and hence $\phi(A \vee B, \ell(\mathbf{H}')) = \mathbf{T}$, as desired. Q.E.D.

The construction just given is obviously closely related to the method used informally in section 1.1 to interpret a model in FC. Under the transformation given by Theorem 1, the two point countermodel to the law of excluded middle in Figure 2 now becomes:

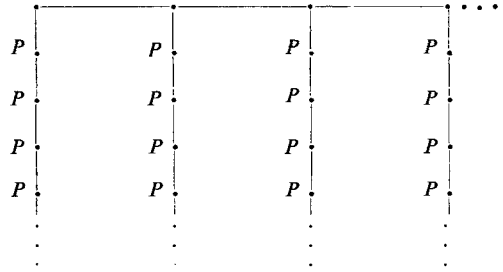

Figure 4a.

It is clear that the model would not essentially change if the infinite vertical branches were reduced to a single point:

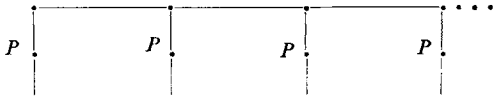

Figure 4b.

Figure 4b is exactly Beth's countermodel in [8] to $P \vee \neg P$.

Since η is a strong Beth model, it is also a model; thus the method allows us to transform a model ϕ on an arbitrary m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ into an "equivalent" model η on a tree m.s. $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$. However, the set $\mathbf{K}'$ will be infinite, even where the original set $\mathbf{K}$ was finite, since the tree $(\mathbf{G}', \mathbf{K}', \boldsymbol{S}')$ has no endpoints. (Also, if $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ is, say, itself a countable but finitary tree m.s., $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$ will not be finitary.) We avoid these difficulties by a modification of the method. First, we observe that the relation $\boldsymbol{R}$, in any q.m.s., may, without loss of generality, be taken to be anti-symmetric (*i.e.* a partial ordering):

LEMMA: *Let* $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ *be any q.m.s., with domain function* $\psi(\mathbf{H})$. *For* $\mathbf{H} \in \mathbf{K}$, *let* $\hat{\mathbf{H}}$ *be the set of* $\mathbf{H}' \in \mathbf{K}$ *such that* $\mathbf{H}\boldsymbol{R}\mathbf{H}'$ *and* $\mathbf{H}'\boldsymbol{R}\mathbf{H}$. *Let* $\hat{\mathbf{K}}$ *be the set of all such* $\hat{\mathbf{H}}$ *for every* $\mathbf{H} \in \mathbf{K}$. *For* $\mathbf{H}, \mathbf{H}' \in \mathbf{K}$, *let* $\hat{\mathbf{H}}\hat{\boldsymbol{R}}\hat{\mathbf{H}}'$ *iff* $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, *and let* $\hat{\psi}(\hat{\mathbf{H}}) = \psi(\mathbf{H})$. *Then* $(\hat{\mathbf{G}}, \hat{\mathbf{K}}, \hat{\boldsymbol{R}})$ *is a q.m.s. with domain function* $\hat{\psi}$. *Moreover, if* ϕ *is a quantificational model on* $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ *and* $\hat{\phi}(P^n, \hat{\mathbf{H}}) = \phi(P^n, \mathbf{H})$ *for any* $\mathbf{H} \in \mathbf{K}$, *then* $\hat{\phi}$ *is a quantificational model on* $(\hat{\mathbf{G}}, \hat{\mathbf{K}}, \hat{\boldsymbol{R}})$ *such that, for any* $\mathbf{H} \in \mathbf{K}$ *and for any formula* A, *relative to any assignment to its free variables,* $\hat{\phi}(A, \hat{\mathbf{H}}) = \phi(A, \mathbf{H})$.

The proof of the lemma is straightforward and is left to the reader.

Theorem 1 (*Second part*): *Let* $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ *be a m.s. such that* $\boldsymbol{R}$ *is a partial ordering. Let* $\boldsymbol{S}$ *be any irreflexive relation such that* $\boldsymbol{R} = \boldsymbol{S}^*$. *Let* $\overline{\mathbf{K}}$ *be the set of all finite non-empty sequences* $\{\mathbf{H}_i\}_{i=1}^n$ *such that* $\mathbf{H}_1 = \mathbf{G}$, *and* $\mathbf{H}_i\boldsymbol{S}\mathbf{H}_{i+1}$ *for every* $i (1 \leq i < n)$. *Let* $\overline{\mathbf{G}}$ *be the sequence whose sole term is* $\mathbf{G}$. *For any* $\overline{\mathbf{H}}_1, \overline{\mathbf{H}}_2 \in \overline{\mathbf{K}}$, *let* $\overline{\mathbf{H}}_1\overline{\boldsymbol{S}}\overline{\mathbf{H}}_2$ *iff* $\overline{\mathbf{H}}_1$ *is the initial segment of* $\overline{\mathbf{H}}_2$ *formed by omitting the last term of* $\overline{\mathbf{H}}_2$. *Let* $\overline{\boldsymbol{R}} = \overline{\boldsymbol{S}}^*$. *Then* $(\overline{\mathbf{G}}, \overline{\mathbf{K}}, \overline{\boldsymbol{R}})$ *is a tree m.s. Moreover, if* ϕ *is a model on* $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$, *and* $\eta(P, \mathbf{H}) = \phi(P, \ell(\overline{\mathbf{H}}))$ (*where* $\ell(\overline{\mathbf{H}})$ *is the last term of* $\overline{\mathbf{H}}$ *for any* $\overline{\mathbf{H}} \in \overline{\mathbf{K}}$), *then* η *is a model on* $(\overline{\mathbf{G}}, \overline{\mathbf{K}}, \overline{\boldsymbol{R}})$ *such that, for any formula* A *of propositional calculus,* $\eta(A, \overline{\mathbf{H}}) = \phi(A, \ell(\overline{\mathbf{H}}))$. *If* $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ *is finite,* $(\overline{\mathbf{G}}, \overline{\mathbf{K}}, \overline{\boldsymbol{R}})$ *is finite also.*

PROOF. It is evident that $(\overline{\mathbf{G}}, \overline{\mathbf{K}}, \overline{\boldsymbol{R}})$ is a tree m.s. Since $\boldsymbol{R}$ is anti-symmetric and $\boldsymbol{S}$ is irreflexive, for every positive n, $\boldsymbol{S}^n$ is irreflexive also; whence, if $\mathbf{K}$ is finite, $\overline{\mathbf{K}}$ will be finite too. If $\overline{\mathbf{H}}_1, \overline{\mathbf{H}}_2 \in \overline{\mathbf{K}}$, and $\overline{\mathbf{H}}_1\overline{\boldsymbol{R}}\overline{\mathbf{H}}_2$, then $\ell(\overline{\mathbf{H}}_1)\boldsymbol{R}\ell(\overline{\mathbf{H}}_2)$: so if $\eta(P, \overline{\mathbf{H}}_1) = \phi(P, \ell(\overline{\mathbf{H}}_1)) = \mathbf{T}$, then, since ϕ is a model, $\eta(P, \overline{\mathbf{H}}_2) = \phi(P, \ell(\overline{\mathbf{H}}_2)) = \mathbf{T}$ also, and thus η is a model. It remains to show, by induction, that, for $\overline{\mathbf{H}} \in \overline{\mathbf{K}}$, and any formula A, $\eta(A, \overline{\mathbf{H}}) = \phi(A, \ell(\overline{\mathbf{H}}))$. The induction step is trivial for $\vee$ and $\wedge$. Suppose $\eta(A \supset B,$

$\overline{\mathbf{H}}) = \mathbf{T}$. Let $\mathbf{H}_1$ be any member of $\mathbf{K}$ such that $\mathbf{H}\boldsymbol{R}\mathbf{H}_1$, where $\mathbf{H} = \ell(\overline{\mathbf{H}})$. Then either $\mathbf{H} = \mathbf{H}_1$, or $\mathbf{H}\boldsymbol{S}^n\mathbf{H}_1$ for some $n > 0$: in either case there exists $\overline{\mathbf{H}}_1 \in \overline{\mathbf{K}}$ such that $\overline{\mathbf{H}}\boldsymbol{R}\overline{\mathbf{H}}_1$ and $\mathbf{H}_1 = \ell(\overline{\mathbf{H}}_1)$. By assumption, either $\eta(A, \overline{\mathbf{H}}_1) = \mathbf{F}$ or $\eta(B, \overline{\mathbf{H}}_1) = \mathbf{T}$, whence, by the induction hypothesis, either $\phi(A, \mathbf{H}_1) = \mathbf{F}$ or $\phi(B, \mathbf{H}_1) = \mathbf{T}$. Since $\mathbf{H}_1$ was arbitrary (subject to $\mathbf{H}\boldsymbol{R}\mathbf{H}_1$, $\phi(A \supset B, \overline{\mathbf{H}}) = \mathbf{T}$. Conversely, suppose $\eta(A \supset B, \overline{\mathbf{H}}) = \mathbf{F}$. Then for some $\overline{\mathbf{H}}_1$ such that $\overline{\mathbf{H}}\boldsymbol{R}\overline{\mathbf{H}}_1$, $\eta(A, \overline{\mathbf{H}}_1) = \mathbf{T}$ and $\eta(B, \overline{\mathbf{H}}_1) = \mathbf{F}$. By the induction hypothesis, $\phi(A, \ell(\overline{\mathbf{H}}_1)) = \mathbf{T}$ and $\phi(B, \ell(\overline{\mathbf{H}}_1)) = \mathbf{F}$; since $\ell(\overline{\mathbf{H}})\boldsymbol{R}\ell(\overline{\mathbf{H}}_1)$, $\phi(A \supset B, \ell(\overline{\mathbf{H}})) = \mathbf{F}$, as desired. The case of $\neg$ is quite similar. Q.E.D.

Notice that the situation contrasts with that in S4, where it is often impossible to replace an arbitrary finite model by an equivalent finite tree model (cf. [2]).

The third part of Theorem 1 extends the procedure for finding a tree model equivalent to an arbitrary model to quantificational models. Here we cannot use the same construction as a tree q. model and as a Beth q. model, as will be seen when we define the latter, in preparation for the fourth part of theorem 1.

THEOREM 1 (*Third part*): *Let* $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ *be a q.m.s. with domain function* $\psi(\mathbf{H})$. ($\boldsymbol{R}$ *need not be anti-symmetric.*) *Let* $\boldsymbol{S}$ *be any relation* (*not necessarily irreflexive*) *such that* $\boldsymbol{R} = \boldsymbol{S}^*$. *Let* ϕ *be a quantificational model on* $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$. *Let* $(\overline{\mathbf{G}}, \overline{\mathbf{K}}, \overline{\boldsymbol{R}})$ *be defined as in the second part of the theorem, and let* $\bar{\psi}(\overline{\mathbf{H}}) = \psi(\ell(\overline{\mathbf{H}}))$. *Let* $\eta(P^n, \overline{\mathbf{H}}) = \varphi(P^n, \ell(\overline{\mathbf{H}}))$ *for each predicate letter* P^n *and each* $\overline{\mathbf{H}} \in \overline{\mathbf{K}}$. *Then* η *is a quantificational model on the q.m.s.* $(\overline{\mathbf{G}}, \overline{\mathbf{K}}, \overline{\boldsymbol{R}})$ *with domain function* $\bar{\psi}$. *Further, relative to a given assignment to the free variables of* A, $\eta(A, \overline{\mathbf{H}}) = \phi(A, \ell(\overline{\mathbf{H}}))$: *in particular,* $\eta(A, \overline{\mathbf{G}}) = \phi(A, \mathbf{G})$.

The proof is left to the reader. Notice that, since $\boldsymbol{S}$ is not required to be irreflexive, it may in particular be $\boldsymbol{R}$ itself: thus $(\overline{\mathbf{G}}, \overline{\mathbf{K}}, \overline{\boldsymbol{R}})$ may be as in the second part of Theorem 1, or may be identical with the Beth model $(\mathbf{G}', \mathbf{K}', \boldsymbol{R}')$ of the first part. As a quantificational model, however, η will not be a Beth quantificational model, to the definition of which we now turn.

Unlike our own models, with their variable domains (a feature we have noted to be essential), the Beth quantificational models are based on a fixed domain $\mathbf{D}$. We define a Beth q.m.s. to be a Beth m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$,

together with a domain **D** with at least one element. A Beth q. *model* η is a binary function $\eta(P^n, \mathbf{H})$, whose value is **T** or **F** when $n = 0$, and is a subset of $\mathbf{D}^n$ for $n \geq 1$. We require, in addition to the conditions (*b*) and (*c*) above on η, the analogues for $n \geq 1$: (b^n). If $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, $\eta(P^n, \mathbf{H}) \subseteq \eta(P^n, \mathbf{H}')$; ($c^n$) if **H** is barred by $\mathbf{B} \subseteq \mathbf{K}$, then

$$\bigcap_{\mathbf{H}' \in \mathbf{B}} \eta(P^n, \mathbf{H}') \subseteq \eta(P^n, \mathbf{H}).$$

For an atomic formula $P^n(x_1, \ldots, x_n)$, define $\eta(P^n(x_1, \ldots, x_n), \mathbf{H}) = \mathbf{T}$, relative to an assignment of $a_1, \ldots a_n \in \mathbf{D}$ to $x_1, \ldots, x_n$, iff $(a_1, \ldots, a_n) \in \eta(P^n, \mathbf{H})$; otherwise, $= \mathbf{F}$. We then define the values for more complex formulae by induction. The inductive clauses for the propositional connectives are as above. Let the formula $A(x_1, \ldots, x_n, y)$ contain only the free variables listed. We define $\eta((y)(A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$, relative to an assignment of $a_i \in \mathbf{D}$ to x_i $(1 \leq i \leq n)$, iff $\eta(A(x_1, \ldots, (x_n, y), \mathbf{H}) = \mathbf{T}$ relative to any assignment of an element $b \in \mathbf{D}$ to y and a_i to x_i; otherwise, $= \mathbf{F}$. Again $\eta((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ when a_i is assigned to x_i iff there is a $\mathbf{B} \subseteq \mathbf{K}$ such that **H** is barred by **B** and for any $\mathbf{H}' \in \mathbf{B}$ there is a $b \in \mathbf{D}$ such that $\eta(A(x_1, \ldots, x_n, y), \mathbf{H}') = \mathbf{T}$ when a_i is assigned to x_i and y is assigned b; otherwise, $= \mathbf{F}$.

Using the inductive clauses and the conditions on atomic formulae, we can prove the analogues of (*b*) and (*c*) for an arbitrary formula *A*, relative to a fixed assignment to its free variables in a Beth quantificational model η. If $\eta(A, \mathbf{H}) = \mathbf{T}$ and $\mathbf{H}\boldsymbol{R}\mathbf{H}'$, $\eta(A, \mathbf{H}') = \mathbf{T}$. If **H** is barred by **B** and $\eta(A, \mathbf{H}') = \mathbf{T}$ for any $\mathbf{H}' \in \mathbf{B}$, then $\eta(A, \mathbf{H}) = \mathbf{T}$.

Suppose we are given a quantificational model ϕ on a m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ such that

$$\mathbf{U} = \bigcup_{\mathbf{H} \in \mathbf{K}} \psi(\mathbf{H})$$

is countable. We will transform ϕ into a Beth quantificational model whose domain **D** is the set **N** of non-negative integers. Let $(\mathbf{G}', \mathbf{K}', \boldsymbol{S}')$ be as above, and $\boldsymbol{R}' = \boldsymbol{S}'^*$. Notice that **N** is a countable union of disjoint countable sets; call these $\mathbf{N}_i (i = 0, \ldots)$. We have a procedure, which, for each $\mathbf{H}' \in \mathbf{K}'$, *generates* certain elements of **N** at $\mathbf{H}'$; the set of elements generated at $\mathbf{H}'$ will be identical with

$$\bigcup_{i=0}^{n} \mathbf{N}_i$$

for some n. Further, if $\mathbf{P}$ is any path in $\mathbf{K}'$, every $p \in \mathbf{N}$ will be generated at some $\mathbf{H}' \in \mathbf{P}$. Further, the procedure will satisfy the condition that if $\mathbf{H}'\boldsymbol{R}'\mathbf{H}''$, every element generated at $\mathbf{H}'$ is also generated at $\mathbf{H}''$. An element generated at $\mathbf{H}'$, but not at its predecessor (if any exists), is said to be *introduced* at $\mathbf{H}'$. Further, any natural number n generated at $\mathbf{H}'$ is assigned a unique element of $\psi(\ell(\mathbf{H}'))$; this element is called $\nu(n, \mathbf{H}')$. The ν-function will satisfy the condition that if n is generated at $\mathbf{H}'$, and $\mathbf{H}'\boldsymbol{R}'\mathbf{H}''$, then $\nu(n, \mathbf{H}') = \nu(n, \mathbf{H}'')$. We give an inductive definition on the tree $(\mathbf{G}', \mathbf{K}', \boldsymbol{S}')$ of a procedure with these properties; at any stage, satisfaction of these properties will be taken to be part of the inductive hypothesis. First, consider the origin $\mathbf{G}'$ of the tree. We generate exactly the elements of $\mathbf{N}_0$ at $\mathbf{G}'$, and we define $\nu(n, \mathbf{G}')$, for $n \in \mathbf{N}_0$, in such a way that $\mathbf{N}_0$ is mapped onto $\psi(\mathbf{G})$. (This is possible since $\psi(\mathbf{G})$ is at most countable. All arbitrary choices can be made precise, if desired, using well-orderings of the denumerable sets $\mathbf{N}$ and $\mathbf{U}$.) Suppose we have defined the set of all integers generated at $\mathbf{H}'$ it is, say,

$$(\mathbf{M} = \bigcup_{i=0}^{m} \mathbf{N}_i)$$

and have defined $\nu(n, \mathbf{H}')$ for each $n \in \mathbf{M}$. Let $\mathbf{H}'\boldsymbol{S}'\mathbf{H}''$. Then *introduce* all elements of $\mathbf{N}_{m+1}$, so that the set of elements generated at $\mathbf{H}''$ is $\mathbf{M} \cup \mathbf{N}_{m+1}$. Define $\nu(n, \mathbf{H}'')$ for $n \in \mathbf{M} \cup \mathbf{N}_{m+1}$ by $\nu(n, \mathbf{H}'') = \nu(n, \mathbf{H}')$ for $n \in \mathbf{M}$, and such that $\nu(n, \mathbf{H}'')$ maps $\mathbf{N}_{m+1}$ onto $\psi(\ell(\mathbf{H}''))$. Then the inductive definition is complete.

We now define a Beth quantificational model η whose domain is $\mathbf{N}$ on the Beth m.s. $(\mathbf{G}', \mathbf{K}' \boldsymbol{S}')$ as follows: If P is a propositional letter, define $\eta(P, \mathbf{H}') = \phi(P, \ell(\mathbf{H}'))$. For an n-adic predicate letter n define $\eta(P^n, \mathbf{H}')$ to be the set of n-tuples $(m_1, \ldots, m_n)$ of natural numbers such that, for every $\mathbf{H}'' \in \mathbf{K}'$ such that $m_1, \ldots, m_n$ are all generated at $\mathbf{H}''$ and $\mathbf{H}'\boldsymbol{R}'\mathbf{H}''$, $(\nu(m_1, \mathbf{H}''), \ldots, \nu(m_n, \mathbf{H}'')) \in \phi(P^n, \ell(\mathbf{H}''))$.

THEOREM 1: (*Fourth part*): *η is a Beth quantificational model on $(\mathbf{G}', \mathbf{K}', \boldsymbol{S}')$ whose domain is $\mathbf{N}$. For any $\mathbf{H}' \in \mathbf{K}'$ and formula $A(x_1, \ldots, x_n)$, whose free variables are exactly those listed, and natural numbers $m_1, \ldots, m_n$, which have been generated at $\mathbf{H}'$, $\eta(A(x_1, \ldots, x_n), \mathbf{H}') = \mathbf{T}$ when $x_1, \ldots, x_n$ are assigned $m_1, \ldots, m_n$, respectively, if and only if $\phi(A(x_1, \ldots, x_n), \ell(\mathbf{H}')) = \mathbf{T}$ when $x_1, \ldots, x_n$ are assigned $\nu(m_1, \mathbf{H}'), \ldots, \nu(m_n,$*

$\mathbf{H}'$), *respectively. In particular* ($n = 0$), *if A is a closed formula*, $\eta(A, \mathbf{H}') = \phi(A, \ell(\mathbf{H}'))$.

PROOF. We show first that η is a Beth quantificational model. Conditions (b) and (b^n) are obvious. Condition (c) is proved as in the first part of the theorem. Condition (c^n) $(n \geq 1)$ is proved as follows: Suppose $\mathbf{H}' \in \mathbf{K}'$ is barred by $\mathbf{B} \subseteq \mathbf{K}'$, and suppose $(m_1, \ldots, m_n)$ is *not* in $\eta(P^n, \mathbf{H}')$. We show that there is an $\mathbf{H}'' \in \mathbf{B}$ such that $(m_1, \ldots, m_n)$ is not in $\eta(P^n, \mathbf{H}'')$. Since $(m_1, \ldots, m_n)$ is *not* in $\eta(P^n, \mathbf{H}')$, there is an $\mathbf{H}'_0 \in \mathbf{K}'$ such that $\mathbf{H}'\boldsymbol{R}'\mathbf{H}'_0$, $m_1, \ldots, m_n$ are all generated at $\mathbf{H}'_0$, and $(v(m_1, \mathbf{H}'_0), \ldots, v(m_n, \mathbf{H}'_0))$ is *not* in $\phi(P^n, \ell(\mathbf{H}'_0))$. As in the first part of this theorem, let $\mathbf{P}$ be the path $\mathbf{P}(\mathbf{H}'_0)$ through $\mathbf{H}'_0$, with the property that, for $\mathbf{H}''$ on the path and $\mathbf{H}'_0\boldsymbol{R}'\mathbf{H}''$, $\ell(\mathbf{H}'_0) = \ell(\mathbf{H}'')$. Then $\mathbf{P}$ intersects $\mathbf{B}$ in an element $\mathbf{H}''$. If $\mathbf{H}''\boldsymbol{R}'\mathbf{H}'_0$, then since clearly $(m_1, \ldots, m_n)$ is not in $\eta(P^n, \mathbf{H}'_0)$, by condition (b^n), it is not in $\eta(P^n, \mathbf{H}'')$. If $\mathbf{H}'_0\boldsymbol{R}'\mathbf{H}''$, then since $\ell(\mathbf{H}'') = \ell(\mathbf{H}'_0)$, and $v(m_i, \mathbf{H}'_0) = v(m_i, \mathbf{H}'')$, we have $((v(m_1, \mathbf{H}''), \ldots, v(m_n, \mathbf{H}'')) \notin \phi(P^n, \ell(\mathbf{H}''))$, so that $(m_1, \ldots, m_n) \notin \eta(P^n, \mathbf{H}'')$, the desired conclusion.

We now prove the assertion in the second sentence of the present *Fourth part* by induction; the third sentence is a special case. Let $A(x_1, \ldots, {}_nx)$ be atomic. If $n = 0$, see the proof of the first part of this theorem. If $n > 0$, write $A(x_1, \ldots, x_n)$ as $P^n(x_1, \ldots, x_n)$. Suppose $m_1, \ldots, m_n$ are all generated at $\mathbf{H}' \in \mathbf{K}'$. Let $\mathbf{H} = \ell(\mathbf{H}')$, and $a_i = v(m_i, \mathbf{H}')$. If $\phi(P^n(x_1, \ldots, x_n), \mathbf{H}) = \mathbf{T}$, when x_i is assigned a_i $(1 \leq i \leq n)$, then $(a_1, \ldots, a_n) \in \phi(P^n, \mathbf{H})$. If $\mathbf{H}'\boldsymbol{R}'\mathbf{H}'_0$ $(\mathbf{H}'_0 \in \mathbf{K}')$, let $\mathbf{H}_0 = \ell(\mathbf{H}'_0)$. Then $\mathbf{H}\boldsymbol{R}\mathbf{H}_0$, hence $a_1, \ldots, a_n \in \psi(\mathbf{H}_0)$. Also $a_i = v(m_i, \mathbf{H}') = v(m_i, \mathbf{H}'_0)$. This shows that $(m_1, \ldots, m_n) \in \eta(P^n, \mathbf{H}')$, hence $\eta(P^n(x_1, \ldots, x_n), \mathbf{H}') = \mathbf{T}$, relative to the assignment of m_i to x_i, as desired. On the other hand, if $\phi(P^n(x_1, \ldots, x_n), \mathbf{H}) = \mathbf{F}$ relative to this assignment, and hence $(a_1, \ldots, a_n) \notin \phi(P^n, \mathbf{H})$, we clearly have $(m_1, \ldots, m_n) \notin \eta(P^n, \mathbf{H}')$, again as desired. The inductive clauses for the propositional connectives are as in the first part of this theorem. Suppose the result proved for $A(x_1, \ldots, x_n, y)$. Again let m_i be assigned to x_i, let $\mathbf{H} = \ell(\mathbf{H}')$, and let $a_i = v(m_i, \mathbf{H}')$ $(i = 1, \ldots, n)$. Let $\phi((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ when x_i is assigned a_i. Then there is a $b \in \psi(\mathbf{H})$ such that $\phi(A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ when in addition y is assigned b. $v(p, \mathbf{H}')$ maps the elements generated at $\mathbf{H}'$ *onto* $\psi(\mathbf{H})$, so let $v(p, \mathbf{H}') = b$, where p is generated at $\mathbf{H}'$. Then, by inductive hypothesis $\eta(A(x_1, \ldots, x_n, y), \mathbf{H}') = \mathbf{T}$ when x_i is assigned m_i $(i = 1, \ldots, n)$ and

y is assigned p; hence $\eta((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H}') = \mathbf{T}$ when x_i is assigned m_i. On the other hand, suppose $\phi((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{F}$ when x_i is assigned a_i. Let $\mathbf{H}'$ be barred by $\mathbf{B} \subseteq \mathbf{K}'$ and let $\mathbf{P}$ be a path in $\mathbf{K}'$ through $\mathbf{H}'$ with the property that for $\mathbf{H}'' \in \mathbf{P}$ and $\mathbf{H}'\boldsymbol{R}'\mathbf{H}''$, $\ell(\mathbf{H}'') = \ell(\mathbf{H}') = \mathbf{H}$. Let $\mathbf{H}''$ be in the intersection of $\mathbf{B}$ and $\mathbf{P}$. Suppose there were a $p \in \mathbf{N}$ such that $\eta(A(x_1, \ldots, x_n, y), \mathbf{H}'') = \mathbf{T}$ when x_i is assigned m_i and y is assigned p. Now it is clear from the definitions that there is an $\mathbf{H}''' \in \mathbf{P}$ such that $\mathbf{H}''\boldsymbol{R}'\mathbf{H}'''$ and p is generated at $\mathbf{H}'''$. Let $v(p, \mathbf{H}''') = b$. Then, since $\mathbf{H}''\boldsymbol{R}'\mathbf{H}'''$, $\eta(A(x_1, \ldots, x_n, y), \mathbf{H}''') = \mathbf{T}$ when x_i is assigned m_i and y is assigned p. Hence by inductive hypothesis, $\phi(A(x_1, \ldots, x_n, y), \ell(\mathbf{H}''')) = \mathbf{T}$ when x_i is assigned a_i and y is assigned b. Further $b = v(p, \mathbf{H}''')$, so $b \in \psi(\ell(\mathbf{H}'''))$. Hence $\phi((\exists y)A(x_1, \ldots, x_n, y), \ell(\mathbf{H}''')) = \mathbf{T}$ when x_i is assigned a_i. But since $\mathbf{H}''' \in \mathbf{P}$, $\ell(\mathbf{H}''')\boldsymbol{R}\mathbf{H}$. Hence $\phi((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ when x_i is assigned a_i, contrary to hypothesis. This *reductio* shows, then, for any $\mathbf{B} \subseteq \mathbf{K}'$ such that $\mathbf{H}'$ is barred by $\mathbf{B}$ there is an $\mathbf{H}'' \in \mathbf{B}$ such that, for every $p \in \mathbf{N}$, $\eta(A(x_1, \ldots, x_n, y), \mathbf{H}'') = \mathbf{F}$ when x_i is assigned m_i and y is assigned p. This in turn shows that $\eta((\exists y)A(x_1, \ldots, x_n, y), \mathbf{H}') = \mathbf{F}$ when x_i is assigned m_i. Thus the case of existential quantification is complete; we now treat universal quantification. Keeping notations as before, suppose $\phi((y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ when x_i is assigned a_i. Suppose $p \in \mathbf{N}$. Let $\mathbf{B} \subseteq \mathbf{K}'$ be the set of all $\mathbf{H}'' \in \mathbf{K}'$ such that $\mathbf{H}'\boldsymbol{R}'\mathbf{H}''$ and p is generated at $\mathbf{H}''$. Let $b = v(p, \mathbf{H}'') \in \psi(\ell(\mathbf{H}''))$. Since $\mathbf{H}'\boldsymbol{R}'\mathbf{H}''$, $\mathbf{H}\boldsymbol{R}\ell(\mathbf{H}'')$, and hence since $\phi((y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{T}$ when x_i is assigned a_i, it follows that $\phi(A(x_1, \ldots, x_n, y), \ell(\mathbf{H}'')) = \mathbf{T}$ when x_i is assigned a_i and y is assigned b. Hence by inductive hypothesis, $\eta(A(x_1, \ldots, x_n, y), \mathbf{H}'') = \mathbf{T}$ when x_i is assigned m_i and y is assigned p. Since $\mathbf{H}'$ is barred by $\mathbf{B}$ and the conclusion held for any $\mathbf{H}'' \in \mathbf{B}$, it follows (since η is a Beth quantificational model), that $\eta(A(x_1, \ldots, x_n, y), \mathbf{H}') = \mathbf{T}$ when x_i is assigned m_i and y is assigned p. Since p was arbitrary, it follows that $\eta((y)A(x_1, \ldots, x_n, y), \mathbf{H}') = \mathbf{T}$ when x_i is assigned m_i, as desired. Suppose finally that $\phi((y)A(x_1, \ldots, x_n, y), \mathbf{H}) = \mathbf{F}$ when x_i is assigned a_i. Then there is an $\mathbf{H}_1 \in \mathbf{K}$ and a $b \in \psi(\mathbf{H}_1)$ such that $\mathbf{H}\boldsymbol{R}\mathbf{H}_1$ and $\phi(A(x_1, \ldots, x_n, y), \mathbf{H}_1) = \mathbf{F}$ when y is assigned b and x_i is assigned a_i. Let $\mathbf{H}_1'$ be formed by adding $\mathbf{H}_1$ as an additional term to the finite sequence $\mathbf{H}'$. Then $\mathbf{H}'\boldsymbol{S}'\mathbf{H}_1'$, $\mathbf{H}_1' \in \mathbf{K}'$, $\ell(\mathbf{H}_1') = \mathbf{H}_1$, and there is a p generated at $\mathbf{H}_1'$ such that $v(p, \mathbf{H}_1') = b$. Then by inductive hypothesis

$\eta(A(x_1, \ldots, x_n, y), \mathbf{H}_1) = \mathbf{F}$ when x_i is assigned m_i and y is assigned p. Hence $\eta((y)A(x_1, \ldots, x_n, y), \mathbf{H}'_1) = \mathbf{F}$ when x_i is assigned m_i; but since $\mathbf{H}'\boldsymbol{R}'\mathbf{H}'_1$, $\eta((y)A(x_1, \ldots, x_n, y), \mathbf{H}') = \mathbf{F}$ relative to this same assignment. This concludes the proof of the theorem. Q.E.D.

The fourth part of Theorem 1 shows how any quantificational model ϕ can be transformed into a Beth quantificational model η. Essentially if we have arrived at a certain position $\mathbf{H}' \in \mathbf{K}'$ and if $\mathbf{H} = \ell(\mathbf{H}')$, the numbers introduced at $\mathbf{H}'$ are "identified" with certain elements of $\psi(\mathbf{H})$ by $v(n, \mathbf{H}')$.

An example, following the spirit though not the letter of the proof of Theorem 1, fourth part, converts the countermodel of section 1.1, Figure 3, for $(x)(P(x) \vee Q). \supset .(x)P(x) \vee Q$ into a corresponding Beth quantificational countermodel in the natural numbers. In Figure 3, there are two evidential situations, $\mathbf{G}$ and $\mathbf{H}$; $\psi(\mathbf{G}) = \{a\}$, $\psi(\mathbf{H}) = \{a, b\}$. As natural numbers are generated, as long as we remain at the evidential situation $\mathbf{G}$, we must "identify" each natural number with a (and therefore give it all properties assigned to a in Figure 3), but if we pass to $\mathbf{H}$, we must "identify" some natural number with b. These considerations lead to the following figure:

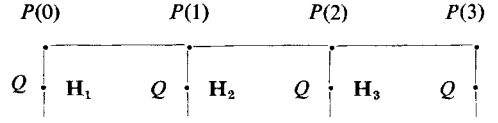

Figure 5.

This is exactly Beth's countermodel to $(x)(P(x) \vee Q). \supset .(x)P(x) \vee Q$. As long as we remain on the horizontal branch but are uncertain that we will continue thereon, we have not established $(x)P(x) \vee Q$; but on the other hand, for each natural number x, either $P(x)$ or Q is eventually established. We have not mechanically applied the proof of Theorem 1 to obtain this model, but instead have reproduced its spirit; in particular, we have introduced a simplification analogous to that required to obtain Figure 4b from Figure 4a.

Notice that Figure 5 can be interpreted in terms of absolutely free choice sequences as follows: Let α be an absolutely free choice sequence on the binary spread. Let $P(x)$ abbreviate $\alpha(x) = 0$, and let Q be $(\exists x)(\alpha(x) = 1)$. Then, if x ranges over the natural numbers, clearly $(\alpha \upharpoonright B)(x)(P(x) \vee Q)$, but $\neg(\alpha \upharpoonright B)((x)P(x) \vee Q)$. And, analogously, as Kreisel

and Dyson (Kreisel [11] and Dyson & Kreisel [9]) have observed, countable Beth quantificational models can always be interpreted thus. So Theorem 1 gives a new intuitive interpretation of our models, in which all quantifiers range over the natural numbers.

Since below we will obtain a completeness theorem for countable quantificational models, and since such models can always be transformed into Beth (q.) models, our completeness results include those of Beth. (Beth required his models to be finitary, but we will show in part II how to obtain finitary Beth models.)

1.3. *Other interpretations of the models*

Sections 1.1 and 1.2 gave interpretations of our models which were intended to accord with the interpretations intuitionists customarily assign to their logical constants. In this section we will give two formal interpretations of the modelling which do not claim any direct intuitionistic content. (Both interpretations are actually direct special cases of the modelling; they simply consider a restricted class of models.) One interpretation is based on provability in formal systems; it was described briefly in [3]. The other is based on Paul Cohen's notion of forcing [5]. The two interpretations are intimately related to each other. This section may be omitted without loss of continuity.

1. *Provability interpretation.* Let $\mathbf{E}_0$ be a formal system, and let $\mathbf{E}$ be an arbitrary extension thereof. Let $\mathbf{K}$ be the set of all such $\mathbf{E}$, and let $\mathbf{E}\boldsymbol{R}\mathbf{E}'$ iff $\mathbf{E}'$ is an extension of $\mathbf{E}$. We define an *atomic formula* P to be a closed wff of $\mathbf{E}_0$. (Note that P need *not* be an atomic formula of $\mathbf{E}_0$.) We can then build non-atomic formulae out of the P's using the connectives $\wedge$, $\supset$, $\neg$, $\vee$. If we define $\phi(P, \mathbf{E}) = \mathbf{T}$ iff P is provable in $\mathbf{E}$ and $\mathbf{F}$ otherwise, then $\phi(P, \mathbf{E})$ is a model on the m.s. $(\mathbf{E}_0, \mathbf{K}, \boldsymbol{R})$. Thus for any complex formula A which is a theorem of the intuitionist propositional calculus, $\phi(A, \mathbf{E}_0) = \mathbf{T}$. If $\mathbf{E}_0$ is elementary number theory Z, and P is Gödel's undecidable formula, then $\phi(P \vee \neg P, \mathbf{E}_0) = \mathbf{F}$; for P is not provable in $\mathbf{E}_0$, but it is provable in certain extensions $\mathbf{E}$. The larger problem, whether Heyting's propositional calculus is complete with respect to this particular choice of $\mathbf{E}_0$, remains open.

To interpret intuitionistic quantification theory in this manner, we must assume that the system $\mathbf{E}_0$ and its extensions have notions of *free*

variables and of *constants*, and that $\mathbf{E}_0$ contains at least one constant. For any $\mathbf{E} \in \mathbf{K}$, let $\psi(\mathbf{E})$ be the set of all constants of $\mathbf{E}$. Then if $\mathbf{E}\boldsymbol{R}\mathbf{E}'$, $\psi(\mathbf{E}) \subseteq \psi(\mathbf{E}')$. For every n, define an n-adic atomic predicate P^n to be a formula of $\mathbf{E}_0$ with n free variables, together with a 1–1 function from the integers $1, \ldots, n$ to the free variables of P^n. The variable assigned by this function to $m (1 \leq m \leq n)$ is called the mth free variable of P^n. Define, for $n \geq 1$, the set $\phi(P^n, \mathbf{E}) \subseteq [\psi(\mathbf{E})]^n$ as follows: An n-tuple $(a_1, \ldots, a_n)$ of constants in $\psi(\mathbf{E})$ is in $\phi(P^n, \mathbf{E})$ iff the result of the simultaneous substitution of $a_i (1 \leq i \leq n)$ for the ith free variable of P^n is a theorem of $\mathbf{E}$. Out of the atomic n-adic predicates (which play the role of the n-adic predicate letters above), we can build more complex formulae using the propositional connectives and the quantifiers. $\phi(P^n, \mathbf{E})$ then becomes an intuitionistic quantificational model.

It is clear that in the preceding $\mathbf{K}$ can be replaced by any subset $\mathbf{K}'$ thereof (*e.g.*, the finitely axiomatizable extensions of $\mathbf{E}_0$). Further, restrictions, such as recursive enumerability, on the notion of formal system, can be removed at will. There is also a more "model-theoretic" variant of the present interpretation of Heyting's predicate calculus, which eliminates the assumption that $\mathbf{E}$ must contain constants. Further, the interpretations can be extended in other directions so as to yield new interpretations of larger parts of intuitionistic mathematics; in particular, we can give an interpretation of FC which leads to a proof that FC is an inessential extension of Heyting's arithmetic[1]). For more on provability interpretations of intuitionistic and modal logics, cf. [3].

2. *Cohen's notion of "forcing."* Let $\mathbf{D}$ be an arbitrary countable infinite set. Let $\mathscr{P} = (\mathscr{P}_0, \mathscr{P}_1)$ be a pair of finite, disjoint subsets of $\mathbf{D}$, and let $\mathbf{K}$ be the set of all such pairs. If $\mathscr{P} = (\mathscr{P}_0, \mathscr{P}_1)$ and $\mathscr{P}' = (\mathscr{P}'_0, \mathscr{P}'_1)$ are in $\mathbf{K}$, then define $\mathscr{P}\boldsymbol{R}\mathscr{P}'$ (or, $\mathscr{P}'$ is an *extension* of $\mathscr{P}$) iff $\mathscr{P}_0 \subseteq \mathscr{P}'_0$ and $\mathscr{P}_1 \subseteq \mathscr{P}'_1$. Further, let $\psi(\mathscr{P}) = \mathscr{P}_0 \cup \mathscr{P}_1$. Now consider a single monadic predicate letter P. For any $\mathscr{P} \in \mathbf{K}$, define $\phi(P, \mathscr{P}) = \mathscr{P}_0$. Let $\mathbf{K}'$ be the set of all $\mathscr{P} \in \mathbf{K}$ such that $\psi(\mathscr{P})$ is non-empty. Then for any $\mathscr{P} \in \mathbf{K}'$, $(\mathscr{P}, \mathbf{K}', \boldsymbol{R})$ is a q.m.s., with the associated domain function ψ. (If we had modified Heyting's predicate calculus so as to admit the empty domain and thus permit $\psi(\mathscr{P})$ to be empty, the rather artificial use of $\mathbf{K}'$ in place

[1]) Kreisel has independently obtained this result using an elimination of free choice sequences by contextual definition.

of **K** could be dropped.) Then ϕ is a model on $(\mathscr{P}, \mathbf{K}', \boldsymbol{R})$, and for any formula A built from P using propositional connectives and quantifiers, the inductive definitions we have given define a truth-value $\phi(A, \mathscr{P}')$, for any $\mathscr{P}' \in \mathbf{K}'$, relative to a fixed assignment of elements of **D** to the free variables of A. If this value is **T**, we say that $\mathscr{P}'$ *forces* A relative to the assignment. (Notice that the value of $\phi(A, \mathscr{P}')$ is clearly independent of the choice of the "designated" element $\mathscr{P}$ of $(\mathscr{P}, \mathbf{K}', \boldsymbol{R})$.)

If $\mathbf{D}'$ is a subset of **D**, we say that $\mathscr{P}'$ *agrees* with $\mathbf{D}'$ iff $\mathscr{P}'_0 \subseteq \mathbf{D}'$ and $\mathscr{P}'_1 \subseteq \mathbf{D} - \mathbf{D}'$. We can say that $\mathbf{D}'$ forces A (relative to a given assignment to the free variables) iff there is a $\mathscr{P}' \in \mathbf{K}'$ which agrees with $\mathbf{D}'$ and forces A. Notice that if $\mathscr{P}'$ and $\mathscr{P}''$ agree with $\mathbf{D}'$, they have a common extension which agrees with $\mathbf{D}'$; thence it easily follows that $\mathbf{D}'$ cannot force a statement together with its negation. Call $\mathbf{D}'$ *generic* iff for every A, and fixed assignment to the free variables thereof, $\mathbf{D}'$ forces either A or $\neg A$. Cohen proves that generic sets exist: Let $\{A_n\}$ be an enumeration of all the ordered couples $A_i = \langle B_i, \theta_i \rangle$ such that B_i is a formula built from P and θ_i is an assignment to its free variables. Define a sequence $\mathscr{P}^n = (\mathscr{P}_0^n, \mathscr{P}_1^n)$ as follows: $\mathscr{P}^0$ is the empty pair. $\mathscr{P}^{n+1}$ is an extension of $\mathscr{P}^n$ which forces B_n (relative to θ_n) if such an extension exists; otherwise it is $\mathscr{P}^n$. Then clearly $\mathscr{P}^{n+1}$ forces B_n or $\neg B_n$ (relative to θ_n), and $\mathbf{D}' = \cup\, \mathscr{P}_0^n$ is generic.

We say $\mathscr{P}'$ *weakly forces* A iff it forces $\neg\neg A$. Noticing (or anticipating) that all provable formulae of Heyting's predicate calculus are valid in our model theory, we get the result that any provable formula A of Heyting's predicate calculus is forced by every $\mathscr{P}' \in \mathbf{K}'$. It is well known that if A does not contain universal quantification and is classically valid, $\neg\neg A$ is provable in Heyting's predicate calculus. Hence, for A classically valid and free of universal quantifiers, A is weakly forced by any $\mathscr{P}' \in \mathbf{K}'$. Further, notice that the formulae forced (weakly forced) by a $\mathscr{P}' \in \mathbf{K}$ are closed under *modus ponens:* A, $A \supset B / B$.

If $\mathbf{D}'$ is generic and forces $\neg\neg A$, it clearly must force A; hence a non-empty, generic $\mathbf{D}'$ forces every classically valid formula not containing universal quantifiers. Cohen has proved an even stronger fact: *If* $\mathbf{D}'$ *is generic and A has no universal quantifiers, then (relative to an assignment to free variables), A is forced by* $\mathbf{D}'$ *if and only if it is true when the existential quantifiers (taken as ranging over* **D**) *and the propositional connectives are interpreted classically, and "*$P(x)$*" is interpreted as "*$x \in \mathbf{D}'$*."* The

assertion is readily proved by induction on the complexity of A. Since, classically speaking, a (x) can always be replaced by $\neg(\exists x)\neg$, the restriction that universal quantifiers be absent is not important.

The definition we have given differs from Cohen's in inessential respects. (It may be closer to a definition given by Feferman, which we have not seen[1]). It is clear that the notion can be extended. For example, we need not deal with a single predicate $P(x)$; we can deal with several such, not all of which need be monadic. The modifications needed for this more general situation should be obvious. Further, we can replace the countable set $\mathbf{D}$ by a set of regular cardinality $\aleph_\alpha$; $\mathbf{K}$ will consist of disjoint pairs of sets of cardinality less than $\aleph_\alpha$.

Cohen's motivation was radically different from ours, but it is clear that his notion is intimately related to our model theory. The "deeper" reasons for this relation may yet be unknown.

It should be noted that Dana Scott had already observed that Cohen's idea was similar to an interpretation conjectured by Kreisel [17]. And indeed, if Kreisel's conjectures prove correct, his interpretation of intuitionism will be closely related to ours.

2. Semantic tableaux

In this section we develop Beth semantic tableaux for intuitionistic logic. The notion developed here is similar to those of [2], [11], which can be read as background if desired. We deal at each stage of the construction with a *system* of *alternative sets* of tableaux; each alternative set is ordered in the form of a tree, and the origin of the tree is called the *main tableau* of the set. We call the tree ordering relation on an alternative set "S"; the smallest reflexive and transitive relation containing "S" is called "R". We can assume, at a given stage of the construction, that each alternative set is diagrammed on a piece of paper; corresponding to the system of all the alternative sets of the stage, we have a *leaflet* of which the separate sheets of paper are *pages*.

Given a formula A of Heyting's predicate calculus, to see whether it is valid we attempt to find a *countermodel* to A. If A has the form $A_1 \wedge \ldots A_m . \supset . B_1 \vee \ldots B_n$, then what we need is a model ϕ, such that relative to some assignment to the free variables of A, $\phi(A_i, \mathbf{G}) = \mathbf{T}$ and

[1]) See note at end of paper.

$\phi(B_j, \mathbf{G}) = \mathbf{F}$, $1 \leq i \leq m$, $1 \leq j \leq n$. We represent the situation by putting $A_1, \ldots, A_m$ on the *left*, and $B_1, \ldots, B_n$ on the *right* of the main tableau of a construction. We continue the construction, which gives a systematic attempt to find a tree countermodel to A, by the following rules, which apply to any tableau of any alternative set of the construction:

Nl. If $\neg A$ appears in the left column of a tableau, put A in the right column of that tableau.

Nr. If $\neg A$ appears in the right column of a tableau t, start out a new tableau t^1, with tSt^1, by putting A on the left of t^1.

$\wedge$l. If $A \wedge B$ appears on the left of a tableau t, put A and B on the left of t.

$\wedge$r. If $A \wedge B$ appears in the right column of a tableau t, there are two alternatives; extend the tableau t either by putting A in the right column or by putting B in the right column. If the tableau t is in an ordered set $\mathscr{S}$, it is clear that at the next stage we have two *alternative* sets, depending on which extension of the tableau t is adopted. Informally speaking, if the original ordered set is diagrammed structurally on a sheet of paper, we copy over the entire diagram twice, in one case putting in addition A in the right column of the tableau and in the other case putting B; the two new sheets correspond to the two new alternative sets. The formal statement is rather messy: Given a tableau t in an alternative set $\mathscr{S}$, if t has $A \wedge B$ on the right, we replace $\mathscr{S}$ by two alternative sets $\mathscr{S}_1$ and $\mathscr{S}_2$, where $\mathscr{S}_1 = \mathscr{S} - \{t\} \cup \{t_1\}$ and $\mathscr{S}_2 = \mathscr{S} - \{t\} \cup \{t_2\}$, and t_1 $[t_2]$ is like t except that in addition it contains A $[B]$ on the right. The tree ordering S_1 of the new set $\mathscr{S}_1$ is precisely the same as S, save that t_1 replaces t throughout; and similarly for the tree ordering S_2 of $\mathscr{S}_2$. (Formally, S_1 agrees with S on $\mathscr{S} - \{t\}$, and, if t' is the predecessor [a successor] of t, then $t'S_1t_1[t_1S_1t']$.) We say $\mathscr{S}$ *splits* into $\mathscr{S}_1$ and $\mathscr{S}_2$. Similar remarks apply to the rule Vl and Pl below.

Vl. If $A \vee B$ appears on the left of t, put either A on the left of t or B on the left of t. (As in the case of $\wedge$r, this splits the set $\mathscr{S}$ containing t into two alternative sets.)

Vr. If $A \vee B$ appears on the right of t, put A and B on the right of t.

Pl. If $A \supset B$ appears on the left of t, either put A on the right of t

or put B on the left. (Thus again the set $\mathscr{S}$ containing t is replaced by two alternative sets.)

Pr. If $A \supset B$ appears on the right of t, start out a new tableau t^1, with A on the left of t^1 and B on the right, such that tSt^1.

For a construction involving quantifiers, we associate, at a given stage of a construction, a set $\psi(t)$ of variables with each tableau t. We start out the definition of $\psi(t)$ by assuming that, at the initial stage of the construction, which starts out with a single tableau t_0, $\psi(t_0)$ consists of a single variable x. At later stages $\psi(t)$ is to be enlarged only as required by the rules Πr and Σl below and the stipulation that tSt^1 is to imply that $\psi(t) \subseteq \psi(t^1)$. We are now in a position to state the rules for quantifiers as follows:

Πl. If $(x)A(x)$ appears on the left of t and y is any variable in $\psi(t)$, put $A(y)$ on the left of t.

Πr. If $(x)A(x)$ appears on the right of t start out a new tableau t^1 with tSt^1. If y is the alphabetically earliest variable which has not yet occurred in any tableau of any alternative set at this stage, put $y \in \psi(t^1)$ and put $A(y)$ on the right of t^1.

Σl. If $(\exists x)A(x)$ appears on the left of a tableau t, and y is the alphabetically earliest variable which has not yet appeared in any tableau of any alternative set at this stage, put $y \in \psi(t)$ and put $A(y)$ of the left of t.

Σr. If $(\exists x)A(x)$ appears on the right of a tableau t, and y is a variable in $\psi(t)$, put $A(y)$ on the right of t.

In addition to the rules we have stated, the following stipulation holds throughout the construction: if t and t^1 are tableaux of some one alternative set, at any given stage, such that tSt^1, and A appears on the left of t, then put A on the left of t^1. Notice that, since the stipulation is to be iterated an arbitrary number of times, it also applies when A is on the left of t and tRt^1.

The relation tSt^1 is to hold in a construction *only* as required by the rules listed above. The rules may be applied in any order, as long as the order stipulated is such that every applicable rule is eventually applied.

A tableau t is called *closed* iff some formula occurs in it on both the left and the right. A set or tree of tableaux is closed iff some tableau in

the set is closed. A system of alternative sets is closed iff every set of the system is closed.

A construction started out by putting A on the right of the main tableau of the construction is called the construction *for* A.

We can place the following restrictions on constructions: A rule is not to be applied to a tableau of a closed set; nor is it to be applied if it is "superfluous" (*e.g.*, A1 is not to be applied if A and B already appear on the left of the tableau t in question).

Let us call an alternative set at any stage of a construction *terminal* iff it is not replaced at any stage of the construction by another set or pair of sets; thus, in particular, every closed set is terminal.

In any construction, let α be some fixed sequence $\mathscr{S}_1, \mathscr{S}_2, \ldots$ of alternative sets such that $\mathscr{S}_1$ is a set at the first stage of the construction and $\mathscr{S}_{i+1}$ is the set or one of the two sets, which, at the $(i+1)$-th stage, replaces $\mathscr{S}_i$; α terminates at $\mathscr{S}_n$ iff $\mathscr{S}_n$ is terminal. (If the construction does not terminate there is at least one infinite such sequence α.) Any tableau t in $\mathscr{S}_1$ or in $\mathscr{S}_{i+1}$ which is not an immediate descendant of any tableau in $\mathscr{S}_i$ is called an initial tableau. Let $\mathbf{K}$ be the set of all sequences τ of tableaux $t_1, t_2, \ldots$ such that t_1 is an initial tableau and t_{i+1} is an immediate descendant of t_i and τ terminates at t_n iff t_n belongs to a terminal set $\mathscr{S}_m$. Let τ_0 be that member of $\mathbf{K}$ whose first term t_1 is in $\mathscr{S}_1$. Let $\tau\rho\tau'$, for τ, τ' in $\mathbf{K}$, iff for some $\mathscr{S}_i$ in α there are terms t, t' of τ, τ' in $\mathscr{S}_i$ such that tRt' (R the ancestral of the tree ordering S). Then, intuitively, $(\tau_0, \mathbf{K}, \rho)$ forms a q.m.s. with domain function

$$\bar{\psi}(\tau) = \bigcup_{t_i \in \tau} \psi(t_i).$$

If a quantificational model ϕ is defined so that, for any sentence letter P, $\phi(P, \tau) = \mathbf{T}$ iff P appears on the left of some t in τ, and, for any predicate letter P^n, $\phi(P^n, \tau)$ is the set of n-tuples $(x_1, \ldots, x_n)$ of variables such that $P^n(x_1, \ldots, x_n)$ appears on the left of some t in τ, then, for every formula B, if B appears on the left of some t in τ, $\phi(B, \tau) = \mathbf{T}$ (relative to the assignment of each free variable in B to itself). Further, the dual law that, for every B, if B appears on the right of some t in τ, then $\phi(B, \tau) = \mathbf{F}$, holds iff α does not terminate in a closed set $\mathscr{S}_n$. Hence, if the construction was a construction for A, this is just the condition under which α provides a countermodel for A.

THEOREM 2: *The construction for A is closed if and only if A is valid.*

The proof, which follows the lines sketched intuitively above, and in addition shows that the alternative sets of the construction for A exhaust the possibilities of finding a countermodel for it, is omitted because it is a routine variation on the proofs of the corresponding theorems of [2] and [16][1]).

3. Completeness theorem

3.1. *Consistency property*

THEOREM 3: *If A is provable in Heyting's predicate calculus, then A is valid.*

This theorem is almost trivial; we need only verify that, in a standard formalization of Heyting's predicate calculus, the axioms are all valid, and the rules preserve validity. Such a verification is left to the reader.

It follows that if A is provable, the construction for A is closed.

3.2. *Completeness property*

We show that every valid formula A is provable by showing that if the construction for A is closed, then A is provable. As in [2] and [16], we do this using a notion of "characteristic formula."

As in [2], define the *rank* of a tableau in a finite tree of tableaux (or, indeed, of a node in any finite tree), as follows: An endpoint of the tree has rank 0. If $\boldsymbol{t}$ is not an endpoint, let $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$ be its successors; then $\text{Rank}(\boldsymbol{t}) = \text{Max}\ \{\text{Rank}\ (\boldsymbol{t}_i)\} + 1$. It is easy to verify that, for any finite tree of tableaux, a unique rank is defined for each tableau of the tree.

[1]) Define A to be *tree valid* iff $\phi\ (A, \mathbf{G}) = \mathbf{T}$ for every model ϕ on a *tree* q.m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$. Then what really is readily proved is that the construction is closed iff A is tree valid. But, by section 1.2 above, validity coincides with tree validity. Alternatively, we can argue as follows without use of section 1.2: Clearly validity implies tree validity, and provability implies validity. The completeness result below shows that tree validity implies provability, so the three notions coincide.

We could have defined a tableau procedure, based on a relation R, which would have been more appropriate to models than to tree models; a reader familiar with [2] will know how this could be carried out.

Notice that, as observed in analogous cases in [2] and [16], the countermodels for non-valid formulae obtained by Theorem 2 from tableaux are always on a *countable* tree q.m.s. $(\mathbf{G}, \mathbf{K}, \boldsymbol{R})$ with a countable set $\mathbf{U}$ of individuals involved. This "Löwenheim-Skolem" result will be used in part II to show that the present completeness results include those of Beth [8].

Given any tableau t in a tree of tableaux, define the following sequence $\{t_i\} : t_0 = t$, t_{j+1} = the predecessor of t_j, if such a predecessor exists, and undefined otherwise. The sequence is clearly finite, and its last term is the origin of the tree. We call it the "path from t back to the origin." The terms of the sequence other than t "come before t" on the tree. For any t on a tree, let $\chi(t)$ be the set of all variables occurring free in t but not in any tableau coming before it.

At any stage of a construction, the tableaux of an alternative set form a finite tree. We define the *characteristic formula* of a tableau t in the set at a given stage by induction on its rank in the set. Given a tableau t, let $A_1, \ldots, A_m[B_1, \ldots, B_n]$ be the formulae occurring on the left [right] of t. Further, let $x_1, \ldots, x_q$ be the elements of $\chi(t)$. (Possibly $q = 0$.) If Rank $(t) = 0$, then the *characteristic formula* of t is defined as $(x_1)\ldots(x_q)\ (A_1 \wedge \ldots A_m . \supset . B_1 \vee \ldots B_n)$; or, if there are no formulae on the left [right] of t, as $(x_1)\ldots(x_q)\ (B_1 \vee \ldots B_n)\ [(x_1)\ldots(x_q)\neg(A_1 \wedge \ldots A_m)]$. If Rank $(t) > 0$, let $t_1, \ldots, t_p$ be the successors of t, and let $C_1, \ldots, C_p$ be the corresponding characteristic formulae. Then the characteristic formula of t is $(x_1)\ldots(x_q)\ (A_1 \wedge \ldots A_m . \supset . B_1 \vee \ldots B_n \vee C_1 \vee \ldots C_p)$; or, if there are no formulae on the left [right] of t, the characteristic formula is $(x_1)\ldots(x_q)\ (B_1 \vee \ldots B_n \vee C_1 \vee \ldots C_p)\ [(x_1)\ldots(x_q)\ (A_1 \wedge \ldots A_m . \supset . C_1 \vee \ldots C_p)]$. The characteristic formula of an alternative set (tree) of tableaux is defined as the characteristic formula of the main tableau of the set. The characteristic formula of the entire system of alternative sets at a given stage of a construction is defined as the conjunction of the characteristic formulae of the alternative sets of the system.

In a natural sense, the present notion of characteristic formula is "dual" to that of [2] and [16]. It may facilitate the reader's comprehension of the notion of characteristic formula if he consults the corresponding treatment of characteristic formulae in [2], [16].

LEMMA: *If A_0 is the characteristic formula of the initial stage of a construction, and B_0 is the characteristic formula of any stage of the construction, then* $\vdash B_0 \supset A_0$.

PROOF. It suffices to show that the characteristic formula of any stage of the construction implies the characteristic formula of the preceding stage. But the characteristic formula of the mth stage has in general the orm $D_1 \wedge \ldots D_j \wedge \ldots D_n$, where the $D_i (1 \leq i \leq n)$ are the characteristic

formulae of the alternative sets of the stage. The rule which is applied and changes the mth stage into the $m + 1$th affects only one alternative set, say with characteristic formula D_j. If the rule is Pl, Λr, or Vl, it will change this set into two distinct alternative sets, with characteristic formulae D'_j and D''_j; we wish to prove, then, $\vdash D_1 \wedge \ldots D'_j \wedge D''_j \wedge \ldots D_n . \supset . D_1 \wedge \ldots D_j \wedge \ldots D_n$. To do this, it suffices to prove $D'_j \wedge D''_j . \supset . D_j$. Similarly, if the rule applied is other than Pl, Λr, or Vl, then D_j is transformed into D'_j; to prove that $\vdash D_1 \wedge \ldots D'_j \wedge \ldots D_n . \supset . D_1 \wedge \ldots D_j \wedge \ldots D_n$, it suffices to prove $\vdash D'_j \supset D_j$. So, when a rule is applied transforming the mth stage of a construction into the $m + 1$th, we need only consider the characteristic formula of the set to which the rule is actually applied.

Suppose, then, a rule (other than Pl, or Λr, or Vl) transforms a set $\mathscr{S}$ with characteristic formula D_j into one with characteristic formula D'_j; we wish to prove $\vdash D'_j \supset D_j$. Let $\boldsymbol{t}$ be the tableau to which the rule is actually applied, and let C be its characteristic formula. Further, let C' be the characteristic formula of the tableau $\boldsymbol{t}'$ into which $\boldsymbol{t}$ is transformed by the given rule. (The rules Nr, Pr and Πr leave $\boldsymbol{t}$ unchanged, appending a new tableau $\boldsymbol{t}^1$. In this case $\boldsymbol{t}'$ will be identical with $\boldsymbol{t}$, but the new characteristic formula C' of $\boldsymbol{t}$ will not be identical with the old one C.) Suppose we can show $\vdash C' \supset C$. Then if $\boldsymbol{t}$ is the main tableau of the set $\mathscr{S}$, we have shown $\vdash D'_j \supset D_j$. Otherwise, let $\boldsymbol{t}_1$ be the predecessor at stage m of $\boldsymbol{t}$, let $\boldsymbol{t}'_1$ be the predecessor at stage $m+1$ of $\boldsymbol{t}'$, and let $C_1[C'_1]$ be the characteristic formula of $\boldsymbol{t}_1[\boldsymbol{t}'_1]$. Then C_1 is a universal quantification (u.q.) of a formula of the form $X . \supset . Y \vee C$, and C'_1 is a u.q. of $X . \supset . Y \vee C'$. Since $\vdash C' \supset C$, clearly $\vdash (X . \supset . Y \vee C') \supset (X . \supset . Y \vee C)$. Applying universal generalization to this last statement, and distributing universal quantifiers across the implication sign, we obtain $\vdash C'_1 \supset C_1$. If $\boldsymbol{t}_1$ is the main tableau of $\mathscr{S}$, then $C'_1 \supset C_1$ is $D'_j \supset D_j$. Otherwise, let $\boldsymbol{t}_2[\boldsymbol{t}'_2]$ be the predecessor of $\boldsymbol{t}_1[\boldsymbol{t}'_1]$, and apply the same reasoning as before. Eventually we will obtain $D'_j \supset D_j$.

Thus in the case of any rule other than Pl, Vl, or Λr, we need only consider the tableau $\boldsymbol{t}$ to which the rule is actually applied, and prove the formula $C' \supset C$ stated above. Notice that in general C, the characteristic formula of $\boldsymbol{t}$, is a u.q. of a certain formula B, and C' is a u.q. of a certain formula B'. If we prove $\vdash B' \supset B$, then by universal generalization and distribution of the quantifiers across the implication sign, we can obtain $C' \supset C$.

Bearing these remarks in mind, we break down the proof into the following cases, depending on the rule applied to obtain the $m + 1$th stage from the mth. We can say a case is "justified," if we have shown, for the case, that $\vdash D_j' \supset D_j$, which usually reduces to $\vdash B' \supset B$. The reader is advised to consult the similar treatments in [2] and [16].

In considering a rule, we will in general assume that the tableau $\boldsymbol{t}$ to which it is applied contains formulae both on the left and the right, and that its characteristic formula is therefore an implication. The cases where the left or right side is empty will be left to the reader.

Case Nl. The characteristic formula of $\boldsymbol{t}$ is a u.q. of $X \wedge \neg A . \supset . Y$; after A has been put on the right, its characteristic formula becomes a u.q. of $X \wedge \neg A . \supset . Y \vee A$. The case is justified by $\vdash X \wedge \neg A . \supset . Y \vee A : \supset : X \wedge \neg A . \supset . Y$.

Case Nr. The characteristic formula of $\boldsymbol{t}$ is a u.q. of $X . \supset . \neg A \vee Y$. When we start out a new tableau $\boldsymbol{t}^1$ with A on the left, and $\boldsymbol{t}S\boldsymbol{t}^1$, the characteristic formula of $\boldsymbol{t}^1$ is $\neg A$ (since $\chi(\boldsymbol{t}^1)$ is empty because any free variable of A already occurs in $\boldsymbol{t}$), and that of $\boldsymbol{t}$ becomes a u.q. of $X . \supset . \neg A \vee Y \vee \neg A$. The case is justified by $\vdash X . \supset . \neg A \vee Y \vee \neg A : \supset : X . \supset . \neg A \vee Y$.

Case Al. Justified by $\vdash X \wedge A \wedge B \wedge A \wedge B . \supset . Y : \supset : X \wedge A \wedge B . \supset . Y$.

Case Ar. Let the characteristic formula of $\boldsymbol{t}$, call it C, be a u.q. of $X . \supset . Y \vee (A \wedge B)$. The rule Ar "splits" $\boldsymbol{t}$ into two alternative tableaux, $\boldsymbol{t}'$ and $\boldsymbol{t}''$, whose characteristic formulae C' and C'' are u.q.'s of $X . \supset . Y \vee (A \wedge B) \vee A$ and $X . \supset . Y \vee (A \wedge B) \vee B$, respectively. Using $\vdash (X . \supset . Y \vee (A \wedge B) \vee A) \wedge (X . \supset . Y \vee (A \wedge B) \vee B) : \supset : X . \supset . Y \vee (A \wedge B)$, and generalizing, and distributing quantifiers, we obtain $\vdash C' \wedge C'' . \supset . C$. If $\boldsymbol{t}$ is the main tableau of the set, this is the desired result $\vdash D_j' \wedge D_j'' . \supset . D_j$. Otherwise, let $\boldsymbol{t}_1$ be the predecessor of $\boldsymbol{t}$. The characteristic formula C_1 of $\boldsymbol{t}_1$ is a u.q. of $X_1 . \supset . Y_1 \vee C$; it is transformed by Ar into two alternative characteristic formulae C_1' and C_1'', which are u.q.'s, respectively, of $X_1 . \supset . Y_1 \vee C'$ and $X_1 . \supset . Y_1 \vee C''$. Using $\vdash C' \wedge C'' . \supset . C$, we easily obtain $\vdash C_1' \wedge C_1'' . \supset . C_1$. Continuing this process along the path from $\boldsymbol{t}$ back to the origin, in a finite number of steps we obtain $\vdash D_j' \wedge D_j'' . \supset . D_j$.

Case Pl. Like Ar, using $\vdash (X \wedge (A \supset B) . \supset . Y \vee A) \wedge (X \wedge (A \supset B) \wedge B . \supset . Y) : \supset : X \wedge (A \supset B) . \supset . Y$.

Case Pr. Let the characteristic formula of t be a u.q. of $X . \supset . Y \vee (A \supset B)$. Pr instructs us to start out a tableau t^1, with A on the left and B on the right, whose characteristic formula is thus $A \supset B$ ($\chi(t^1)$ being empty). Then the characteristic formula of t is transformed into a u.q. of $X . \supset . Y \vee (A \supset B) \vee (A \supset B)$, and $\vdash X . \supset . Y \vee (A \supset B) \vee (A \supset B) : \supset : X . \supset . Y \vee (A \supset B)$ justifies the case.

Case Vl. Like Λr, using $\vdash (X \wedge (A \vee B) \wedge A . \supset . Y) \wedge (X \wedge (A \vee B) \wedge B . \supset . Y) : \supset : (X \wedge (A \vee B) . \supset . Y)$.

Case Vr. Justified by $\vdash X . \supset . Y \vee (A \vee B) \vee A \vee B : \supset : X . \supset . Y \vee (A \vee B)$.

Case Σl. If t has as characteristic formula C, a u.q. of $X \wedge (\exists x) A(x) . \supset . Y$, after application of Σl, t is transformed into t^1, whose characteristic formula C' is a u.q. of $X \wedge (\exists x) A(x) \wedge A(a) . \supset . Y$. Since a is a new variable not previously introduced, $a \in \chi(t^1)$. Thus, we can take C' to be a u.q. of $(a) (X \wedge (\exists x) A(x) \wedge A(a) . \supset . Y)$. So $\vdash (a) (X \wedge (\exists x) A(x) \wedge A(a) . \supset . Y) : \supset : X \wedge (\exists x) A(x) . \supset . Y$ justifies the case.

Case Σr. Justified by $\vdash X . \supset . Y \vee (\exists x) A(x) \wedge A(a) : \supset : X . \supset . Y \vee (\exists x) A(x)$.

Case Πl. Justified by $\vdash X \wedge (x) A(x) \wedge A(a) . \supset . Y : \supset : X \wedge (x) A(x) . \supset . Y$.

Case Πr. The characteristic formula of t is a u.q. of $X . \supset . Y \vee (x) A(x)$. Πr instructs us to start out a new tableau t^1, with tSt^1, and with $A(a)$ on the right, where a has not previously been used. Then $\chi(t^1) = \{a\}$, since a is the only free variable of t^1 which does not occur in t. Hence the characteristic formula of t^1 is $(a) A(a)$, and the characteristic formula of t is transformed into a u.q. of $X . \supset . Y \vee (x) A(x) \vee (a) A(a)$. So $\vdash X . \supset . Y \vee (x) A(x) \vee (a) A(a) : \supset : X . \supset . Y \vee (x) A(x)$ justifies the case.

Finally, we must justify the rule stipulating that if a formula A appears on the left of a tableau t, and tSt^1, we must put A on the left of t^1. This is justified by $X \wedge A . \supset . Y \vee ((X' \wedge A) \supset Y') : \supset : X \wedge A . \supset . Y \vee (X' \supset Y')$. The lemma is proved.

THEOREM 4: *If A is valid, then A is provable in Heyting's predicate calculus.*

PROOF. We can assume A has no free variables. Since A is valid, the

construction for A is closed. Then there is a stage at which each alternative set is closed; let the characteristic formula of that stage be $D_1 \wedge \ldots D_n$, where the D_j's are the characteristic formulae of the alternative sets of the stage. By the lemma, $D_1 \wedge \ldots D_n . \supset . A$ (since A is the characteristic formula of the initial stage). So it suffices to show D_j for each j. The alternative set whose characteristic formula is D_j, being closed, contains a closed tableau $\boldsymbol{t}$. Then $\boldsymbol{t}$ contains a formula B on both sides, so its characteristic formula C is a u.q. of $X \wedge B . \supset . Y \vee B$. Clearly $\vdash C$. If $\boldsymbol{t}$ is the main tableau of the set, this is D_j. Otherwise, let $\boldsymbol{t}_1$ be the predecessor of $\boldsymbol{t}$. Then the characteristic formula C_1 of $\boldsymbol{t}_1$ is a u.q. of $X' . \supset . Y' \vee C$. Clearly $\vdash C_1$. Continuing in this manner, we are driven back along the path from $\boldsymbol{t}$ to the origin until we obtain $\vdash D_j$. Q.E.D.

REMARK. The theorem gives a finitary proof that if the construction for A is closed, $\vdash A$. We could have proved it alternatively by showing that the tableau procedure is equivalent to a standard Gentzen formulation of Heyting's system. Of course the theorem and proof apply to the propositional calculus, even though the proof was carried out for the predicate calculus.

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