Subject 24.244. Modal Logic. Answers to the first p-set.

1. **Consider SC connectives "NAND" and "XOR," with the following truth tables:**

φ	ψ	(φ NAND ψ)	(φ XOR ψ)
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	F

a) **Given a sentence containing only the connective "NAND" that is logically equivalent to "(P XOR Q," or explain why there can be no such connective.**

P	Q	((P	NAND	(Q NAND Q))	NAND	((P NAND P)	NAND	Q))
T	T	T	T	F	F	F	T	T
T	F	T	F	T	T	F	T	F
F	T	F	T	F	T	T	F	T
F	F	F	T	T	F	T	T	F

b) **Given a sentence containing only the connective "XOR" that is logically equivalent to "(P NAND Q," or explain why there can be no such connective.**
No such sentence. Each atomic sentences as a "F" at the bottom line of the truth table. Any formula obtained by putting "XOR" between two formulas with "F" at the last line of the truth table will have "F" at the bottom line. So any formula constructed using no connective other than "XOR" will have "F" at the bottom line of the truth table. "P NAND Q" has "F" at the bottom line of the truth table. So no formula formed using no connective other than "XOR" has the same truth table as "NAND."

2. **Use the Compactness Theorem to show that, for Γ and Δ sets of SC sentences, the following are equivalent:**
$\Gamma \cup \Delta$ is inconsistent.
There is a sentence φ such that Γ implies φ and Δ implies $\sim \varphi$.
($\Rightarrow$) If $\Gamma \cup \Delta$ is inconsistent, the Compactness Theorem tells us that there is a finite inconsistent subset Ω of $\Gamma \cup \Delta$. Assuming that $\Omega \cap \Gamma$ is nonempty, take φ to be the conjunction of the members of $\Omega \cup \Gamma$. Γ implies φ. $\{\varphi\} \cup \Delta$ is inconsistent, so Δ implies $\sim \varphi$.

If, on the other hand, $\Omega \cap \Gamma$ is empty, Ω is an inconsistent subset of Δ. So Δ is inconsistent, so it implies every formula. Take an atomic formula α. Γ implies $(\alpha \vee \sim \alpha)$, and Δ implies $\sim (\alpha \vee \sim \alpha)$.

3. **Within the version of the sentential calculus in which the atomic sentences are uppercase letters from the English alphabet, with or without Arabic numeral subscripts, let us say that a set S of complete stories is *closed* iff there is a set of sentences Γ such that S = {complete stories that include Γ}.**

a) **True or false? Explain your answer: The intersection of two closed sets of complete stories is always closed.**
True. Suppose C and D are closed. There are sets Γ and Δ of sentences such that C = {complete stories that contain Γ} and D = {complete sentences Δ}. C $\cap$ D = {complete stories that contain $\Gamma \cup \Delta$.

b) **True or false? Explain your answer: The union of two closed sets of complete stories is always closed.**
True. Same notation as part a). The same formulas will be in every complete story that contains $\{(\alpha \vee \sim \alpha)\}$ (where α is an atomic sentence) as are true in every complete story that contains $\varnothing$. So

it will do no harm to assume that Γ and Δ are both nonempty. Let Ω be the set of all sentences $(\gamma \vee \delta)$ with $\gamma \in \Gamma$ and $\delta \in \Delta$. We want to show that $C \cup D$ = {the set of complete stories that contain Ω}.

If w is in C and γ is in Γ, γ in in w. If γ in in Γ and δ is in Δ, any complete story that contains γ contains $(\gamma \vee \delta)$. So every member of Ω is in w, and $C \subseteq$ {complete stories that contain Ω}. By the same argument, $D \subseteq$ {complete stories that include Ω}.

If w is in neither C nor D, there is an elements γ of Γ that isn't in w, and an element δ of Δ that isn't in w. $\sim \gamma$ and $\sim \delta$ are both in w, so $\sim (\gamma \vee \delta)$ is in w, so $(\gamma \vee \delta)$ isn't in w and w isn't in {complete stories that include Ω}.

c) **Let's say a set of complete stories is *clopen* if it and its complete are both closed. Show that a set of complete stories is clopen iff there is a sentence φ with S = {complete stories that include φ}.**
(⇒) Suppose that S is the set of complete stories that contain Γ and the complement of S, $W \sim S$, is the set of complete stories that contain Δ. Since no complete story is in both S and $W \sim S$, $\Gamma \cup \Delta$ is inconsistent. By problem 2, there is a sentence φ implied by Γ whose negation is implied by Δ. If w is in S, w contains Γ and so it contains φ. If w isn't in S, w contains Δ, so it contains $\sim \varphi$, so it doesn't contain φ. So S = {complete stories that contain φ}.

(⇐) If S = {complete stories that contain φ}, S = {complete stories that contain {φ}}, so it's closed. The complement of S = {complete stories that don't contain φ} = {complete stories that containg $\sim \varphi$} = {complete stories that contains {$\sim \varphi$}}, so it's closed too. So S is clopen.

d) **True or false? Explain your answer: The complement of a closed set of complete stories is always closed.**
False. Let Γ be the set of all the atomic sentences, and let w be the unique complete story that includes Γ. {w} is closed. If the complement of {w} is also closed, then by part c) there is a sentence φ such that {w} = {complete stories that include φ}. Since there are infinitely many atomic sentences, we can find an atomic sentence α that doesn't occur in φ. Let v be the complete story that contains $\sim \alpha$ and contains all the atomic sentences other than α. Since w and v agree in the values they assign to all the atomic sentences that occur in φ, φ is in one of them iff it's in the other. So φ is in v. But that's impossible, since v is in the complement of {w}/

4. **Would any of the answers to problem 3 have changed if we were talking about the language whose atomic sentences are the 26 uppercase English letters, without the numerical subscripts? Explain your answer.**
The answers to a), b), and c) don't depend in any way on how many atomic sentenes are in the language. d) is another story. In the language with 26 atomic sentences, there are 2^{26} complete stories, and each of them has a unique state description. Give a set S of complete stories, let φ be the disjunction of the state descriptions of the members of S (or if S in empty, let φ be "$(A \wedge \sim A)$"). If w is in S, the state description of w is one of the disjuncts of φ, so $\varphi \in w$. If w isn't in S, the state description of w is incompatible with all the disjuncts of φ, so the state description of w is incompatible with φ, so $\varphi \notin w$. So S = {complete stories that include φ}. By part c), S is clopen. Every set of world is both open and closed.

5. **For each of the following sentences, either give a derivation in S5 or present a simple Kripke model in which it's false. In doing the derivations, you may use the derived rules from the $\vee$ lecture notes.**

a) $((\Box P \vee \Box Q) \leftrightarrow \Box(P \vee Q))$

Consider a model with two worlds, @ and w, and in interpretation that assigns True to <"P",@> and <"Q",w> and False to <"P",w> and <"Q",@>. The "$(P \vee Q)$" is true in both world, so "$\Box(P \vee Q)$" is true in <W,I,@>. Neither "$\Box P$" nor "$\Box Q$" is true in <W,I,@>.

b) $((\Box P \wedge \Box Q) \leftrightarrow \Box(P \wedge Q))$

1.	$(P \rightarrow (Q \rightarrow (P \wedge Q)))$	(Taut)
2.	$\Box(P \rightarrow (Q \rightarrow (P \wedge Q)))$	(Nec) 1
3.	$(\Box P \rightarrow (\Box Q \rightarrow \Box(P \wedge Q)))$	(K) 2
4.	$((P \wedge Q) \rightarrow P)$	(Taut)
5.	$\Box((P \wedge Q) \rightarrow P)$	(Nec) 4
6.	$(\Box(P \wedge Q) \rightarrow \Box P)$	(K) 5
7.	$((P \wedge Q) \rightarrow Q)$	(Taut)
8.	$\Box((P \wedge Q) \rightarrow Q)$	(Nec) 7
9.	$(\Box(P \wedge Q) \rightarrow \Box Q)$	(K) 8
10.	$((\Box P \wedge \Box Q) \leftrightarrow (\Box P \wedge \Box Q))$	(TC)3, 6, 9

c) $((\Diamond(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q))$

Take a model with two worlds, @ and w, with "P" true in @ but not in w, and with "Q" true in neither word. "$(P \rightarrow Q)$" is true in w, so "$\Diamond(P \rightarrow Q)$" is true in @. "$\Diamond P$" is true in @. "$\Diamond Q$" isn't true in @. So "$((\Diamond P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q))$" is false in <W,I,@>.

d) $((\Diamond P \rightarrow \Diamond Q) \rightarrow \Diamond(P \rightarrow Q))$

1.	$(\sim (P \rightarrow Q) \rightarrow P)$	(Taut)
2.	$\Box(\sim (P \rightarrow Q) \rightarrow P)$	(Nec) 1
3.	$(\Box\sim (P \rightarrow Q) \rightarrow \Box P)$	(K) 2
4.	$(\Box P \rightarrow P)$	(T)
5.	$(\Box\sim P \rightarrow \sim P)$	(T)
6.	$(\sim \sim \Box \sim (P \rightarrow Q) \rightarrow \sim \Box \sim P)$	(TC) 3, 4,5
7.	$(\sim \Diamond(P \rightarrow Q) \rightarrow \Diamond P)$	(TC) 6, Def. of "$\Diamond$"
8.	$(\Diamond P \rightarrow \Diamond(P \rightarrow Q))$	(TC) 7
9.	$(\sim (P \rightarrow Q) \rightarrow \sim Q)$	(Taut)
10.	$\Box(\sim (P \rightarrow Q) \rightarrow \sim Q)$	(Nec) 9
11.	$(\Box \sim (P \rightarrow Q) \rightarrow \Box \sim Q)$	(K) 10
12.	$(\sim \sim \Box\sim (P \rightarrow Q) \rightarrow \sim \sim \Box \sim Q)$	(TC) 11
13.	$(\sim \Diamond (P \rightarrow Q) \rightarrow \sim \Diamond Q)$	(TC) 10, Def. of "$\Diamond$"
14.	$(\sim \Diamond(P \rightarrow Q) \rightarrow \sim (\Diamond P \rightarrow \Diamond Q))$	(TC) 7, 13
15.	$((\Diamond P \rightarrow \Diamond Q) \rightarrow \Diamond(P \rightarrow Q))$	(TC) 14

e) $(\Box(P \vee (\Diamond Q \vee \Box R)) \leftrightarrow (\Box P \vee(\Diamond Q \vee \Box R))$

First we work on the left-to-right direction:

1.	$((P \vee (\Diamond Q \vee \Box R)) \rightarrow (\sim(\Diamond Q \vee \Box R) \rightarrow P))$	(Taut)
2.	$\Box((P \vee (\Diamond Q \vee \Box R)) \rightarrow (\sim(\Diamond Q \vee \Box R) \rightarrow P))$	(Nec) 1
3.	$(\Box(P \vee (\Diamond Q \vee \Box R)) \rightarrow (\Box\sim(\Diamond Q \vee \Box R) \rightarrow \Box P))$	(K) 2

If we can show that "$\sim(\Diamond Q \vee \Box R)$" implies "$\Box\sim(\Diamond Q \vee \Box R)$," this will get us within a single application of (TC) of where we want to go. We can accomplish this by showing that "$(\sim\Diamond Q \wedge \sim\Box R)$" implies "$(\Box\sim\Diamond Q \wedge \Box\sim\Box R)$." We start the process by showing that "$\sim \Diamond Q$" implies "$\Box\sim\Diamond Q$," using (4):

4.	$(\Box\sim Q \rightarrow \Box \Box\sim Q)$	(Taut), Def. of "$\Diamond$"
5.	$(\sim\Diamond Q \rightarrow \Box\Box\sim Q)$	(TC)4, Def. of "$\Diamond$
6.	$(\Box\sim Q \rightarrow \sim\Diamond Q)$	(Taut, Def. of "$\Diamond$"

7. $\Box(\Box{\sim}Q \rightarrow {\sim}\Diamond Q)$ (Nec) 6

8. $(\Box\Box{\sim}Q \rightarrow \Box{\sim}\Diamond Q)$ (K) 7

9. $({\sim}\Diamond Q \rightarrow \Box{\sim}\Diamond Q)$ (TC) 5, 8

Now we show that "~□R" implies "□~□R, using (5). We'll do this by showing "(~□R ↔ ◊~R)," and applying (Subs).

10. $(\Diamond{\sim}R \rightarrow \Box\Diamond{\sim}R)$ (5)

11. $(R \rightarrow {\sim}{\sim}R)$ (Taut)

12. $\Box(R \rightarrow {\sim}{\sim}R)$ (Nec)11

13. $(\Box R \rightarrow \Box{\sim}{\sim}R)$ (K) 12

14. $(\Diamond{\sim}R \rightarrow {\sim}\Box R)$ (TC) 13, Def. of "◊"

15. $({\sim}{\sim}R \rightarrow R)$ (Taut)

16. $\Box({\sim}{\sim}R \rightarrow R)$ (Nec), 15

17. $(\Box{\sim}{\sim}R \rightarrow \Box R)$ (K), 16

18. $({\sim}\Box R \rightarrow \Diamond{\sim}R)$ (TC) 17, Def. of "◊"

19. $({\sim}\Box R \leftrightarrow \Diamond{\sim}R)$ (TC) 14, 18

20. $({\sim}\Box R \rightarrow \Box{\sim}\Box R)$ (Subs) 10, 19

We put the pieces together to get the left-to-right:

21. $({\sim}\Diamond Q \rightarrow ({\sim}\Box R \rightarrow {\sim}(\Diamond Q \vee \Box R)))$ (Taut)

22. $\Box({\sim}\Diamond Q \rightarrow ({\sim}\Box R \rightarrow {\sim}(\Diamond Q \vee \Box R)))$ (Nec) 21

23. $(\Box{\sim}\Diamond Q \rightarrow (\Box{\sim}\Box R \rightarrow \Box{\sim}(\Diamond Q \vee \Box R)))$ (K) 22

24. $({\sim}\Diamond Q \rightarrow ({\sim}\Box R \rightarrow \Box{\sim}(\Diamond Q \vee \Box R)))$ (TC) 9, 20, 23

25. $({\sim}(\Diamond Q \vee \Box R) \rightarrow \Box{\sim}(\Diamond Q \vee \Box R))$ (TC)24

26. $(\Box(P \vee (\Diamond Q \vee \Box R)) \rightarrow ({\sim}(\Diamond Q \vee \Box R) \rightarrow \Box P))$ (TC) 3, 25

27. $(\Box(P \vee (\Diamond Q \vee \Box R)) \rightarrow (\Box P \vee (\Diamond Q \vee \Box R)))$ (TC) 26

We're done with that half. Now we show the right-to-left direction, by showing that each of the there disjuncts of the right-hand side imples the left-hand side. First "□P":

28. $(P \rightarrow (P \vee (\Diamond Q \vee \Box R)))$ (Taut)

29. $\Box(P \rightarrow (P \vee (\Diamond Q \vee \Box R)))$ (Nec) 28

30. $(\Box P \rightarrow \Box(P \vee (\Diamond Q \vee \Box R)))$ (K) 29

Next "◊Q":

31. $(\Diamond Q \rightarrow (P \vee (\Diamond Q \vee \Box R)))$ (Taut)

32. $\Box(\Diamond Q \rightarrow (P \vee (\Diamond Q \vee \Box R)))$ (Nec) 31

33. $(\Box\Diamond Q \rightarrow \Box(P \vee (\Diamond Q \vee \Box R)))$ (K) 32

34. $(\Diamond Q \rightarrow \Box\Diamond Q)$ (5)

35. $(\Diamond Q \rightarrow \Box(P \vee (\Diamond Q \vee \Box R)))$ (TC) 33, 34

Now "□R":

36. $(\Box R \rightarrow (P \vee (\Diamond Q \vee \Box R)))$ (Taut)

37. $\Box(\Box R \rightarrow (P \vee (\Diamond Q \vee \Box R)))$ (Nec) 36

38. $(\Box\Box R \rightarrow \Box(P \vee (\Diamond Q \vee \Box R)))$ (K) 37

39. $(\Box R \rightarrow \Box\Box R)$ (4)

40. $(\Box R \rightarrow \Box(P \vee (\Diamond Q \vee \Box R)))$ (TC) 38, 39

Finally, we assemble all the pieces:

41. $((\Box P \vee (\Diamond Q \vee \Box R)) \rightarrow \Box(P \vee (\Diamond Q \vee \Box R)))$ (TC) 30, 35, 40

42. $(\Box(P \vee (\Diamond Q \vee \Box R)) \leftrightarrow (\Box P \vee (\Diamond Q \vee \Box R))$ (TC) 27, 41

f) $(\Box\Diamond\Box(P \leftrightarrow Q) \leftrightarrow \Diamond\Box\Diamond(P \leftrightarrow Q))$

Take a model with two worlds, @ and w. "P" is true in both worlds, whereas "Q" is true in only @. Since "(P ↔Q)" is true in @, "◊(P ↔Q)" is true in both worlds. So "□ ◊(P ↔ Q)" is true in both worlds. So "◊ □ ◊(P ↔ Q)" is true in both world. "(P ↔ Q)" is false in w. So "□(P ↔ Q)" is false in both worlds. So "◊□(P ↔ Q)" is false in both worlds. So "□◊□(P ↔ Q)" is false in both worlds. So the biconditional is false in both worlds.