Intuitionistic Sentential Calculus

Intuitionism is a revolutionary program for rebuilding the foundations of mathematics. Initiated by L.E.J Brouwer,[1] its central idea is that mathematics is a creation of the human mind. Since mathematical entities don't have any kind of autonomous existence, they don't have any properties beyond the properties we build into them by our theorizing. There isn't anything to being a real number beyond having the properties with which we mentally endow the real numbers in our theorizing. If there is a question we can't answer, that we are unable to answer no matter how assiduously we attempt to answer it, then the question doesn't have an answer, for there is no objective mathematical reality to supply answers we can't provide.

Is there an n such that there is no sequence of exactly n 7s in the decimal expansion of π? Nobody knows, and there is no reason to expect that anyone ever will or can know. Can we at least say this much: Either there is an n such that there is a sequence of exactly n 7s in the decimal expansion of π or, for every n, there is a sequence of exactly n 7s in the decimal expansion of π. Classical mathematics says that of course we can, since the law of the excluded middle, which permits us to assert sentences of the form $(\varphi \vee \sim \varphi)$, is a law of logic. Intuitionists say that, if the issue is indeed undecidable (it could happen that there's a proof we don't know about) then the disjunction isn't true because neither disjunct is true.

Truth, as intuitionists conceive it, has a lot in common with truth in a work of fiction, but there's a crucial difference. The author of *Beowolf* doesn't tell us the color of King Hrothgar's eyes, and since Hrothgar is a fictional character, he doesn't have any attributes beyond those given to him by the poet. So within the fictional world of *Beowolf*, the statement that Hrothgar had blue eyes in neither true of false. Nevertheless, we can say this much: Within the story, either Hrothgar had blue eyes or he didn't have blue eyes. According to intuitionists, mathematical statements have a tacit "within mathematics" in the same way that statements like "Beowolf tore off Grendel's arm" need to be understood as really saying, "within the *Beowolf* story, Beowolf tore off Grendel's arm." Here's the key difference: Within the *Beowolf* story, either Hrothgar's eyes were blue or they were not. According to intuitionists, we cannot say: Within mathematics, either there is an n for which there is no sequence of exactly n 7s in π or there is no such n. We can't say that because saying it would blur the boundary between mathematics and fiction. Mathematics may be about a subject that is a human creation, but its theorems are objectively true.

Intuitionists have their own logic, which is more restrictive than classical logic. Here is an example, due to Dummett,[2] of reason that is legitimate classically but not intuitionistically:

Theorem. There are irrational numbers b and c such that b^c is rational.

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[1]"On the Significance of the Principle of Excluded Middle" in Jean van Heijenoort, ed., *From Frege to Gödel* (Harvard University Press, 1967), pp. 334-345.

[2]*Elements of Intuitionism* 2nd ed. (Oxford University Press, 2000), p. 6.

Proof: There are two cases:
Case 1. $\sqrt{2}^{\sqrt{2}}$ is rational. Set b = c = $\sqrt{2}$.
Case 2. $\sqrt{2}^{\sqrt{2}}$ is irrational. Set b = $\sqrt{2}^{\sqrt{2}}$ and c = $\sqrt{2}$. Then $b^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, which is rational.⊠

To classical mathematicians, this is a perfectly good proof, but intuitionists will reject it. Since we have no method of testing whether $\sqrt{2}^{\sqrt{2}}$ is rational, we have no assurance that either case obtains.

Intuitionistic thinking has been extended beyond mathematics by tying the doctrine to verificationism. Sentences don't have an intrinsic meaning. Somehow, they are given their meanings by us, the speakers. How? Verificationists say that the meaning of a sentence is fixed by our procedures for testing whether the sentence obtains. If we are unable, even if we have unbounded time and resources, to determine whether a sentence is true, then the sentence in neither true nor false, and intuitionistic logic should apply.[3]

Rules of Intuitionistic Sentential Calculus

Arend Heyting[4] gave a system of intuitionistically sound rules for the sentential calculus. The main difference between Heyting's rules and the classical rules is the absence of the law of the excluded middle. Heyting extended that system to include the intuitionistic predicate calculus, but we won't follow him there. The distinctive feature of the intuitionistic predicate calculus is its rejection of nonconstructive existence proofs, that is, proof that purport to establish the existence of a mathematical entity satisfying a particular description without being able to identify the entity. Thus intuitionists will reject our classical proof that there exist irrational numbers b and c such that b^c is rational. We identified two candidates, but didn't single one out.

The system we'll look at here isn't Heyting's system but a system of "natural deduction," which aims to stick as closely as possible to the ways good reasoners reason informally. We'll follow Gerhard Gentzen,[5] though not in exact detail.

Intuitionists identify truth with idealized provability. A conjunction is intuitionistically true iff it's provable, which happens iff both conjuncts are provable, which happens iff both conjuncts are true.

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[3]See Michael Dummett, "Truth" in *Truth and Other Enigmas* (Harvard University Press, 1978), pp. 1-24.

[4]*Intuitionism: An Introduction*, 2nd revised ed. (North-Holland, 1966), pp. 97-101.

[5]"The Calculus of Natural Deduction" in "Investigations into Logical Deduction" in M. E. Szabo, ed., *The Collected Papers of Gerhard Gentzen* (North-Holland, 1969), pp. 74-80.

A disjunction is true intuitionistically iff one both disjuncts are true intuitionistically, which happens iff one or both disjuncts are provable. So instances of excluded middle aren't necessarily intuitionistically true.

To say that a conditional is true intuitionistically means that there is a partial proof which, if tacked onto a proof of the antecedent, would yield a proof the the consequent.

To say that a negation is true means that a contradiction can be derived from the negatum.

Classically, we could treat "→" as defined: $(\varphi \rightarrow \psi) =_{\text{Def}} (\sim \varphi \vee \psi)$. If you had a proof either of $\sim \varphi$ or ψ, you could combine it with a proof of φ, if you had one, to get a proof of ψ. So $(\sim \varphi \vee \psi)$ intuitionistically implies $(\varphi \rightarrow \psi)$. But having a partial proof that you could tack onto a hypothetical proof of φ to get a proof of ψ doesn't provide us either with a refutation of φ or a proof of ψ. So the equivalence $((\varphi \rightarrow \psi) \leftrightarrow (\sim \varphi \vee \psi))$ is intuitionistically invalid.

The logical terms of the intuitionistic formal language will be "$\vee$," "$\wedge$," "$\rightarrow$," "$\sim$," and "$\perp$." "$\perp$" is a special atomic sentence that's always false.

$\vdash_{\text{Int}}$ is the smallest relation relating sets of sentences to sentences that meets the following conditions:

Identity If $\gamma \in \Gamma$, then $\Gamma \vdash_{\text{Int}} \gamma$.
Transitivity If $\Gamma \vdash_{\text{Int}} \delta$ for each member δ of Δ and $\Delta \vdash_{\text{Int}} \varphi$, then $\Gamma \vdash_{\text{Int}} \varphi$.
"$\wedge$"*-introduction* $\{\varphi,\psi\} \vdash_{\text{Int}} (\varphi \wedge \psi)$.
"$\wedge$"*-elimination* $\{(\varphi \wedge \psi)\} \vdash_{\text{Int}} \varphi$ and $\{(\varphi \wedge \psi)\} \vdash_{\text{Int}} \psi$.
"$\vee$"*-introduction* $\{\varphi\} \vdash_{\text{Int}} (\varphi \vee \psi)$, and $\{\psi\} \vdash_{\text{Int}} (\varphi \vee \psi)$.
Proof by cases If $\Gamma \cup \{\varphi\} \vdash_{\text{Int}} \theta$ and $\Gamma \cup \{\psi\} \vdash_{\text{Int}} \theta$, then $\Gamma \cup \{(\varphi \vee \psi)\} \vdash_{\text{Int}} \theta$.
Modus ponens $\{\varphi\ (\varphi \rightarrow \psi)\} \vdash_{\text{Int}} \psi$.
Conditional proof If $\Gamma \cup \{\varphi\} \vdash_{\text{Int}} \psi$, $\Gamma \vdash_{\text{Int}} (\varphi \rightarrow \psi)$.
Ex contradictione quodlibet $\{\perp\} \vdash_{\text{Int}} \chi$, any χ.
Law of contradiction $\{\varphi, \sim\varphi\} \vdash_{\text{Int}} \perp$.
Intuitionistic reductio If $\Gamma \cup \{\varphi\} \vdash_{\text{Int}} \perp$, $\Gamma \vdash_{\text{Int}} \sim \varphi$.

If $\Gamma \vdash_{\text{Int}} \varphi$, then there is a finite subset Γ_{Fin} of Γ such that $\Gamma_{\text{Fin}} \vdash_{\text{Int}} \varphi$. Transitivity, proof by cases, conditional proof, and intuitionistic reductio are indirect rules. The others are direct rules.If $\Omega \vdash_{\text{Int}} \varphi$, then there is a finite Ω_{Fin} of Ω for which we can obtain $\Omega_{\text{Fin}} \vdash_{\text{Int}} \varphi$ at the end of a finite sequence, each member of which is either an application of a direct rule or obtained from earlier members of the sequence by an indirect rules, and where each set that occurs to the left of "$\vdash_{\text{Int}}$" is finite. In this situation, we'll say that φ is "intuitionistically derivable" from Ω.

Intuitionistic Kripke Models

Kripke invented a way of applying possible-world semantics to the study of intuitionistic logic. A "possible world" is an possible epistemic state, a way that, as far as we can tell at a particular time, the mathematical situation might be. If we're in epistemic state w, we can't go beyond w by logic alone, but we might be able to employ some mathematical proof or mathematical insight to reach a new epistemic state v that goes beyond w. We'll say that v is accessible from w. We are talking about the epistemic states of idealized agents, who always reason rigorously, and who are fully confident in the rigor of their reasoning. So if v is accessible from w, anything that you know in w will still be known by you in v.

An *intuitionistic model* is an transitive, reflexive model <W,R,I,@> that obeys the condition that, if I(α,w) = T and wRv, I(α,v) = T. We're using our old notation, but there's a danger. We say that, from an intuitionistic point of view, an undecidable statement is neither true nor false because it's neither provable nor refutable, but when we write "I(α,w) = F," we don't mean that α can be refuted but that it hasn't been proved. We should think of "I(α,w) = T" and "I(α,w) = F" as "α can be proved in w" and "α can't be proved in w," leaving truth and falsity out of it.

We define what it is for a sentence to be intuitionistically true in an intuitionistic model.

An atomic sentence α is intuitionistically true in <W,R,I,w> iff I(α,w) = T.

A conjunction is intuitionistically true in <W,R,I,w> iff both conjuncts are intuitionistically true in <W,R,I,w>

A disjunction is intuitionistically true in <W,R,I,w> iff one or both disjuncts are intuitionistically true in <W,R,I,w>.

A conditional (φ → ψ) is intuitionistically true in <W,R,I,w> iff the ψ is intuitionistically true in every model <W,R,I,v> in which wRv and φ is intuitionistically true.

⊥ isn't intuitionistically true in <W,R,I,w>.

A negation ~ φ is intuitionistically true in <W,R,I,w> iff there is no world v with wRv such that φ is true in <W,R,I,v>.

The rules were carefully designed to ensure that we have the following property, which we prove by an easy induction on the complexity of θ:

> **Monotony**. If θ is intuitionistically true in <W,R,I,w> and wRv, then θ is intuitionistically true in <W,R,I,v>.

Soundness and Completeness

It's straightforward to verify, by examining the rules one by one, that if $\Gamma \vdash_{Int} \chi$, then χ is intuitionistically true in every intuitionistic model in which all the members of Γ are intuitionistically true. We want to show the converse, that if $\Gamma \nvdash_{Int} \chi$, then there is an intuitionistic model in which all the members of Γ are intuitionistically true but χ isn't intuitionistically true.

Our proof will imitate the completeness proofs we had earlier, but it won't duplicate it. Before we took our worlds to be complete stories, but now we don't want our worlds to be complete. We'll want them to leave some questions unsettled.

The canonical frame for intuitionistic logic is the triple <W,R,I> where:

> W = {sets of formulas w with the property that, for some formula ζ, w $\nvdash_{Int} \zeta$, but for any set s of formulas that properly includes w, s $\vdash_{Int} \zeta$.
>
> wRv iff $w \subseteq v$.
>
> $I(\alpha,w) = T$ iff $\alpha \in w$.

The center of the proof is the following:

> **Truth Lemma.** A formula is intuitionistically true in a world in the canonical frame iff it's an element of the world.

Proof: By induction on the complexity of formulas.

Case 1. For an atomic formula α, α is intuitionistically true in w iff $I(\alpha,w) = T$, which happens iff $\alpha \in w$.

Case 2. We know from transitivity that any formula derivable from a world in an element of the world. It follows that, for given φ and ψ, $(\varphi \wedge \psi)$ is an element of a world w iff φ and ψ are both elements of w. $(\varphi \wedge \psi)$ is intuitionistically true in w iff φ and ψ are both intuitionistically true in w iff φ and ψ are both elements of w (by inductive hypothesis) iff $(\varphi \wedge \psi)$ is an element of w.

Case 3. If a disjunction $(\varphi \vee \psi)$ is intuitionistically true in w, either φ or ψ is intuitionistically true in w. So either φ or ψ is an element of w, by inductive hypothesis. So $(\varphi \vee \psi)$ is an element of w.

If $(\varphi \vee \psi)$ isn't intuitionistically true in w, neither φ nor ψ is intuitionistically true in w. By inductive hypothesis, neither φ nor ψ in an element of w. There is a formula ζ such that ζ isn't derivable from w but ζ is derivable from any set of formulas that properly includes w. So ζ

is derivable from $w \cup \{\varphi\}$ and also derivable from $w \cup \{\psi\}$. It follows by proof by cases that ζ is derivable from $w \cup \{(\varphi \vee \psi)\}$. So $(\varphi \vee \psi)$ isn't in w.

Case 4. If $(\varphi \rightarrow \psi)$ isn't in w, $w \nvdash (\varphi \rightarrow \psi)$. By conditional proof, $w \cup \{\varphi\} \nvdash \psi$. Take v to be a maximal set of formulas containing $w \cup \{\varphi\}$ from which ψ isn't derivable. v is a world accessible from w containing φ and not containing ψ. By inductive hypothesis, φ is intuitionistically true in v and ψ isn't intuitionistically true in v. So $(\varphi \rightarrow \psi)$ isn't intuitionistically true in w.

If $(\varphi \rightarrow \psi)$ is in w, it's in any world accessible from w. If v is a world accessible from w in which φ is intuitionistically true, the inductive hypothesis tells us that φ is in v as well as $(\varphi \rightarrow \psi)$. By modus ponens, ψ is in v, so by inductive hypothesis, ψ is intuitionistically true in v. It follows that $(\varphi \rightarrow \psi)$ is intuitionistically true in w.

Case 5. For any world w, there is a sentence ζ that isn't derivable from w. This implies, because of *ex contradictione quodlibet*, that $\perp$ can't be in w. Moreover, $\perp$ isn't intuitionistically true in w.

Case 6. If $\sim \varphi$ is intuitionistcally true in w, then there isn't any world accessible from w in which φ is intutionistically true. So, by inductive hypothesis, there isn't any world that contains both w and φ. If $\perp$ weren't derivable from $w \cup \{\varphi\}$, we could find a maximal set including $w \cup \{\varphi\}$ from which $\perp$ isn't derivable, thereby finding a world containing $w \cup \{\varphi\}$. So $\perp$ is derivable from $w \cup \{\varphi\}$. By intuitionistic reductio, $\sim \varphi$ is derivable from w, so $\sim \varphi$ is an element of w.

If $\sim \varphi$ is in w, then any world accessible from w in which φ is intuitionistically true would be, by inductive hypothesis, a world that contains both φ and $\sim \varphi$, and so, according to the law of contradiction, it would contain $\perp$. But that can't happen, by case 5. So there is no world accessible from w in which φ is intutionistically true. Hence $\sim \varphi$ is intuitionistically true in w. ⊠

Now that we have the truth lemma, the completeness theorem is easy. If χ isn't derivable from Γ, we can find a maximal set w containing Γ from which χ isn't derivable. w is a world in which, according the the truth lemma, all the members of Γ are intuitionistically true and χ is not.⊠

Now that we have the completenenss theorem, we can show that sentences aren't derivable by finding intuitionist models in which they aren't true. For example, take a model with two worlds @ and w with R= {<@,@>, <@,w>, <w,w>}. Let "P" be true in w only. Then neither "P" nor "$\sim$ P" is intuitionistically true in @. So "$(P \vee \sim P)$" isn't intuitionistically true in @.

Double negation introduction – the rule that lets us derive $\sim \sim\varphi$ from $\{\varphi\}$ – is derivable, but double negation elimination is not. From $\{\varphi,\sim \varphi\}$ you can derive $\perp$ by the law of contradiction. By intuitionist reductio, you can derive $\sim\sim\varphi$ from $\{\varphi\}$. On the other hand, if we have two worlds @ and w with $R = \{\langle @,@\rangle,\langle @,w\rangle,\langle w,w\rangle\}$ and we have "P" true in w only, "~P" won't be true in either world, so "~ ~ P" will be true in@, even though "P" is not.

De Morgan's law tells us the $(\sim\varphi \vee \sim\psi)$ is equivalent to $\sim(\varphi \wedge \psi)$. The left-to-right direction is valid. If there is no world accessible from @ in which φ is intuitionistically true, there is no world accessible from @ in which $(\varphi \wedge \psi)$ is intuitionistically true. If there is no world accessible from @ in which ψ is intuitionistically true, there is no world accessible from @ in which $(\varphi \wedge \psi)$ is intuitionistically true. So if $(\sim\varphi \vee \sim\psi)$ is intuitionistically true in @, either $\sim\varphi$ is intuitionistically true in @, in which case $\sim(\varphi \wedge \psi)$ is intuitionistically true in @, or $\sim\psi$ is intuitionistically true in @, in which case again $\sim(\varphi \wedge \psi)$ is true in @.

The right-to-left direction of de Morgan's law isn't intuitionistically valid. Take a model with three worlds, @, w, and v, with $R = \{\langle @,@\rangle,\langle @,w\rangle,\langle @,v\rangle,\langle w,w\rangle,\langle v,v\rangle\}$ and with "P" true in w only and "Q" true in v only. "~P" isn't intuitionistically true in @ and "~Q" isn't intuitionistically true in @, so "$(\sim P \vee \sim Q)$" isn't intuitionistically true in @, even though "$\sim(P \wedge Q)$" is intuitionistically true in @.

One form of contraposition tells us that, if from $\Gamma \cup \{\varphi\}$ you can derive ψ, then from $\Gamma \cup \{\sim\psi\}$ you can derive $\sim\varphi$. This form is intuitionistically valid. Suppose all the members of $\Gamma \cup \{\sim\psi\}$ are intuitionistically true in w. Take v accessible from w. Since all the members of Γ are intuitionistically true in v and ψ isn't intuitionistically true in v, φ must not be intuitionistically true in v. This holds for any world v accessible from w, so $\sim\varphi$ is intuitionistically true in w.

The other form of contrapositions tells us that, if from $\Gamma \cup \{\sim\psi\}$ you can derive $\sim\varphi$, then from $\Gamma \cup \{\varphi\}$ you can derive ψ. This form isn't intuitionistially valid. From $\{\sim P\}$ you can derive $\sim\sim\sim P$ by double negation introduction, but you can't derive P from $\{\sim\sim P\}$.

The rule we have is intuitionistic reductio. Classical reductio tell us that, if you can derive $\perp$ from $\Gamma \cup \{\sim\varphi\}$, then from Γ you can derive φ. This isn't intuitionisticaly valid. From $\{\sim\sim P, \sim P\}$ you can derive $\perp$, but from $\{\sim\sim P\}$ you can't derive P.

Classical Natural Deduction

We can get a sound and complete system of natural deduction for classical logic by adding the double negation elimination to the intuitionistic rules. Let $\vdash_{\text{Class}}$ be the smallest relation that includes the intuitionistic rules and also the following:

Double negation elimination $\{\sim\sim\varphi\} \vdash_{\text{Class}} \varphi$.

It's clear by examining the rules, including DNE, that if $\Gamma \vdash_{\text{Class}} \chi$, then every complete storythat contains Γ contains χ. We want to show that converse, that if $\Gamma \nvdash_{\text{Class}} \chi$, then there is a complete story that includes Γ but not χ.

We begin by getting the following derived rule for $\vdash_{\text{Class}}$:

Exhaustion If $\Gamma \cup \{\varphi\} \vdash_{\text{Class}} \chi$ and $\Gamma \cup \{\sim \varphi\} \vdash_{\text{Class}} \chi$, then $\Gamma \vdash_{\text{Class}} \chi$.

If $\Gamma \cup \{\varphi\} \vdash_{\text{Class}} \chi$, then $\Gamma \cup \{\varphi, \sim \chi\} \vdash_{\text{Class}} \bot$, by the law of contradiction and transitivity. By intuitionistic reductio, $\Gamma \cup \{\sim \chi\} \vdash_{\text{Class}} \sim \varphi$. If $\Gamma \cup \{\sim \varphi\} \vdash_{\text{Class}} \chi$, $\Gamma \cup \{\sim \chi\} \vdash_{\text{Class}} \chi$, and hence $\Gamma \cup \{\sim \chi\} \vdash_{\text{Class}} \bot$. By intuitionistic reductio, $\Gamma \vdash_{\text{Class}} \sim\sim \chi$. By DNE, $\Gamma \vdash_{\text{Class}} \chi$.

If we carry out the canonical frame construction using $\vdash_{\text{Class}}$ instead of $\vdash_{\text{Int}}$, the exhaustion rule assures us that every world in a complete story. If w is a world, there is a sentence ζ such that w is a maximal set from which ζ is not derivable. Take any sentence χ. If neither χ nor $\sim \chi$ is in w, then ζ is derivable from $w \cup \{\chi\}$ and also derivable form $w \cup \{\sim \chi\}$. Exhaustion tells us that ζ is derivable from w, which is impossible.

If χ isn't classically derivable from Γ, then there is a world in the canonical frame that contains all the members of Γ and excludes χ. That world is a complete story.

Intuitionistic Logic and S4

In 1933, Gödel[6] gave a method for translating an intuitionistic language into a classically interpreted modal language. He defined a translation Tr with the property that a sentence χ is an intuitioinist consequence of a set of sentences Γ if and only if $\text{Tr}(\chi)$ is an S4-consequence of the images under Tr of Γ:

$\text{Tr}(\alpha) = \Box\alpha$, for α atomic.
$\text{Tr}(\varphi \vee \psi) = \Box(\text{Tr}(\varphi) \vee \text{Tr}(\psi))$.
$\text{Tr}(\varphi \wedge \psi) = \Box(\text{Tr}(\varphi) \wedge \text{Tr}(\psi))$.
$\text{Tr}(\varphi \rightarrow \psi) = \Box(\text{Tr}(\varphi) \rightarrow \text{Tr}(\psi))$.
$\text{Tr}(\bot) = \Box\bot$.
$\text{Tr}(\sim \varphi) = \Box \sim \text{Tr}(\varphi)$.

Theorem.[7] $\Gamma \vdash_{\text{Int}} \chi$ iff $\text{Tr}(\chi)$ is an S4-consequence of $\{\text{Tr}(\gamma): \gamma \in \Gamma\}$.

Proof: (⇐) Given an intuitionistic frame <W,R,I>, we have the following:

[6]"An Interpretation of the Intuitionistic Propositional Calculus," *Collected Works*, vol. 1 (Oxford University Press, 1986), pp 290-301.

[7]Gödel proved the left-to-right direction and conjectured the right-to-left, which was proved by J. C. C. Mckinsey and Alfred Tarski, "The Algebra of Topology," *Annals of Mathematics* 45 (1944): 141-191.

Lemma. A formula θ is intuitionistically true in <W,R,I,w> iff $Tr(\theta)$ is classically true in <W,R,I,w>.

Proof: By induction on the complexity of θ. I'll only go through one case here, the case where θ is a disjunction $(\varphi \vee \psi)$. We have:

$(\varphi \vee \psi)$ is intuitionistically true in w
iff either φ or ψ is intuitionistically true in w
iff either $Tr(\varphi)$ or $Tr(\psi)$ is classically true in w [by inductive hypothesis]
iff $(Tr(\varphi) \vee Tr(\psi))$ is classically true in w
(*) iff $\Box(Tr(\varphi) \vee Tr(\psi))$ is classically true in w
iff $Tr(\varphi \vee \psi)$ is classically true in w.

We get (*) by noting that $Tr(\varphi)$ and $Tr(\psi)$ both begin with "$\Box$"s, say $\varphi = \Box\mu$ and $\psi = \Box\nu$, and observing that the following is a theorem of S4:

$((\Box\mu \vee \Box\nu) \leftrightarrow \Box(\Box\mu \vee \Box\nu))$.⊠

If $\Gamma \nvdash_{Int} \chi$, there is an intuitionistic model <W,R,I,@> in which all the members of Γ are intuitionistically true and χ is intuitionistically false. <W,R,I,@> is an transitive, reflexive model in which all the members of $\{Tr(\gamma): \gamma \in \Gamma\}$ are classically true and $Tr(\chi)$ is classically false.

(⇒). Given an S4-model <W,R,I,@> define an intuitionistic interpretation <W,R,I*,@> by stipulating that $I^*(\alpha,w) = T$ iff $I(\alpha,v) = T$ for all v with wRv. We have the following, by induction on the complexity of θ:

Lemma. θ is intuitionistically true in <W,R,I*,w> iff $Tr(\theta)$ is classically true in <W,R,I,w>.

If $Tr(\chi)$ isn't an S4-consequence of $\{Tr(\gamma): \gamma \in \Gamma\}$, there is a transitive, reflexive model <W,R,I,@> in which all the members of $\{Tr(\gamma): \gamma \in \Gamma\}$ are classically true and $Tr(\chi)$ is classically false. In <W,R,I*,@>, all the members of Γ are intuitionistically true and χ is intuitionistically false.⊠

Since there is an algorithm for testing whether an inference with finitely many premises is modally valid in S4, we have this:

Corollary. There is an algorithm for testing whether an inferene with finitely many premises is intuitionistically valid.