JOHN P. BURGESS

BASIC TENSE LOGIC

1 WHAT IS TENSE LOGIC?

We approach this question through an example:

(1) *Smith*: Have you heard? Jones is going to Albania!
Smythe: He won't get in without an extra-special visa. Has he remembered to apply for one?
Smith: Not yet, so far as I know.
Smythe: Then he'll have to do so soon.

In this bit of dialogue the argument, such as it is, turns on issues of temporal order. In English, as in all Indo-European and many other languages, such order is expressed in part through changes in verb-form, or tenses. How should the logician treat such tensed arguments?

A solution that comes naturally to mathematical logicians, and that has been forcefully advocated in [Quine, 1960], is to regiment ordinary tensed language to make it fit the patterns of classical logic. Thus Equation 1 might be reduced to the quasi-English Equation 1 below, and thence to the 'canonical notation' of Equation 3:

(2) Jones/visits/Albania at some time later than the present.

At any time later than the present, if Jones/visits/Albania then, then at some earlier time Jones/applies/for a visa.

At no time earlier than or equal to the present it is the case that Jones/applies/for a visa.

Therefore, Jones/applies/for a visa at some time later than the present.

(3)
$$\exists t(c < t \wedge P(t))$$
$$\forall t(c < t \wedge P(t) \rightarrow \exists u(u < t \wedge Q(u)))$$
$$\neg\exists t((t < c \vee t = c) \wedge Q(t))$$
$$\therefore \exists t(c < t \wedge Q(t)).$$

Regimentation involves introducing quantification over instants $t, u, \ldots$ of time, plus symbols of the present instant c and the earlier- later relation $<$. Above all, it involves treating such a linguistic item as 'Jones is visiting Albania' *not* as a complete sentence expressing a proposition and having a truth-value, to be symbolised by a sentential variable $p, q, \ldots$, but rather as a predicate expressing a property on instants, to be symbolised by a one-place predicate variable $P, Q, \ldots$. Regimentation has been called *detensing*

D. Gabbay and F. Guenthner (eds.),
Handbook of Philosophical Logic, Volume 7, 1–42.

since the verb in, say, 'Jones/visits/Albania at time t', written here in the grammatical present tense, ought really to be regarded as *tenseless*; for it states not a present fact but a timeless or 'eternal' property of the instant t. Bracketing is one convention for indicating such tenselessness. The knack for regimenting or detensing, for reducing something like Equation 1 to something like Equation 3, is easily acquired. The analysis, however, cannot stop there. For a tensed argument like that above must surely be regarded as an *enthymeme*, having as unstated premises certain assumptions about the structure of Time. Smith and Smythe, for instance, probably take it for grated that of any two distinct instants, one is earlier than the other. And if this assumption is formalised and added as an extra premise, then Equation 3, invalid as it stands, becomes valid.

Of course, it is the job of the cosmologist, not the logician, to judge whether such an assumption is physically or metaphysically correct. What *is* the logician's job is to formalise such assumptions, correct or not, in logical symbolism. Fortunately, most assumptions people make about the structure of Time go over readily into first- or, at worst, second-order formulas.

1.1 Postulates for Earlier-Later

(B0)	Antisymmetry	$\forall x \forall y \neg(x < y \wedge y < x)$
(B1)	Transitivity	$\forall x \forall y \forall z(x < y \wedge y < z \rightarrow x < z)$
(B2)	Comparability	$\forall x \forall y(x < y \vee x = y \vee y < x)$
(B3)	(a) Maximum	$\exists x \forall y(y < x \vee y = x)$
	(b) Minimum	$\exists x \forall y(x < y \vee x = y)$
(B4)	(a) No Maximals	$\forall x \exists y(x < y)$
	(b) No Minimals	$\forall x \exists y(y < x)$
(B5)	Density	$\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y))$
(B6)	(a) Successors	$\forall x \exists y(x < y \wedge \neg \exists z(x < z \wedge z < y))$
	(b) Predecessors	$\forall x \exists y(y < x \wedge \neg \exists z(y < z \wedge z < x))$
(B7)	Completeness	$\forall U((\exists x U(x) \wedge \exists x \neg U(x) \wedge$
		$\forall x \forall y(U(x) \wedge$
		$\wedge \neg U(y) \rightarrow x < y)) \rightarrow$
		$(\exists x(u(x) \wedge$
		$\wedge \forall y(x < y \rightarrow \neg U(y))) \vee$
		$\exists x(\neg U(x) \wedge$
		$\wedge \forall y(y < x \rightarrow U(y))))$
(B8)	Wellfoundedness	$\forall U(\exists x U(x) \rightarrow \exists x(U(x) \rightarrow$
		$\wedge \forall y(y < x \rightarrow \neg U(y)))$
(B9)	(a) Upper Bounds	$\forall x \forall y \exists z(x < z \wedge y < z)$
	(b) Lower Bounds	$\forall x \forall y \exists z(z < x \wedge z < y)$.

For more on the development of the logic of time as a branch of applied first- and second-order logic, see [van Benthem, 1978].

The alternative to regimentation is the development of an autonomous *tense logic* (also called *temporal* logic or *chronological* logic), first undertaken in [Prior, 1957] (though several precursors are cited in [Prior, 1967]). Tense logic takes seriously the idea that items like 'Jones is visiting Albania' are already complete sentences expressing propositions and having truth-values, and that they should therefore by symbolised by sentential variables $p, q, \ldots$. Of course, the truth-value of a sentence in the present tense may well differ from that of the corresponding sentence in the past or future tense. Hence, tense logic will need some way of symbolising the relations between sentences that differ only in the tense of the main verb. At its simplest, tense logic adds for this purpose to classical truth-functional sentential logic just two one-place connectives: the future-tense or 'will' operator F and the past-tense or 'was' operator P. Thus, if p symbolises 'Jones is visiting Albania', then Fp and Pp respectively symbolise something like 'Jones is sooner or later going to visit Albania' and 'Jones has at least once visited Albania'. In reading tense-logical symbolism aloud. F and P may be read respectively as 'it will be the case that' and 'it was the case that'. Then $\neg F \neg$, usually abbreviated G, and $\neg P \neg$, usually abbreviated H, may be read respectively as 'it is always going to be the case that' and 'it has always been the case that'. Actually, for many purposes it is preferable to take G and H as primitive, defining F and P as $\neg G \neg$ and $\neg H \neg$ respectively. Armed with this notation, the tense-logician will reduce Equation 1 above to the stylised Equation 1.1 and then to the tense-logical Equation 5:

(4) Future-tense (Jones visits Albania)

Not future-tense (Jones visits Albania and not past-tense (Jones applies for a visa)).

Not past-tense (Jones applies for a visa) and not Jones applies for a visa.

Therefore, future-tense (Jones applies for a visa)

(5)
$$\begin{array}{l} Fp \\ \neg F(p \wedge \neg Pq) \\ \neg Pq \wedge \neg q \\ \therefore Fq. \end{array}$$

Of course, we will want some axioms and rules for the new temporal operators F, P, g, H. All the axiomatic systems considered in this survey will share the same standard format.

1.2 Standard Format

We start from a stock of sentential *variables* $p_0, p_2, p_2, \ldots$, usually writing p for p_0 and q for p_1. The (well-formed) *formulas* of tense logic are built up from the variables using negation ($\neg$), and conjunction ($\wedge$), and the strong future (G) and strong past (H) operators. The *mirror image* of a formula is the result of replacing each occurrence of G by H and vice versa. Disjunction ($\vee$), material conditional ($\rightarrow$), material biconditional ($\leftrightarrow$), constant true ($\top$), constant false ($\bot$), weak future (F), and weak past (P) can be introduced as abbreviations.

As *axioms* we take all substitution instances of truth-functional tautologies. In addition, each particular system will take as axioms all substitution instances of some finite list of extra axioms, called the *characteristic* axioms of the system. As *rules* of inference we take Modus Ponens (MP) plus the specifically tense-logical:

Temporal Generalisation(TG): From α to infer $G\alpha$ and $H\alpha$

The *theses* of a system are the formulas obtainable from its axioms by these rules. A formula is *consistent* if its negation is not a thesis; a set of formulas is *consistent* if the conjunction of any finite subset is. These notions are, of course, relative to a given system.

The systems considered in this survey will have characteristic axioms drawn from the following list:

1.3 Postulates for a Past-Present-Future

(A0)	(a)	$G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq)$	(b)	$H(p \rightarrow q) \rightarrow (Hp \rightarrow Hq)$
	(c)	$p \rightarrow GPp$	(d)	$p \rightarrow HFp$
(A1)	(a)	$Gp \rightarrow GGp$	(b)	$Hp \rightarrow HHp$
(A2)	(a)	$Pp \wedge Fq \rightarrow F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)$		
	(b)	$Pp \wedge Pq \rightarrow P(p \wedge Pq) \vee P(p \wedge q) \vee P(Pp \wedge q)$		
(A3)	(a)	$G\bot \vee FG\bot$	(b)	$H\bot \vee PH\bot$
(A4)	(a)	$Gp \rightarrow Fp$	(b)	$Hp \rightarrow Pp$
(A5)	(a)	$Fp \rightarrow FFp$	(b)	$Pp \rightarrow PPp$
(A6)	(a)	$p \wedge Hp \rightarrow FHp$	(b)	$p \wedge Gp \rightarrow PGp$
(A7)	(a)	$Fp \wedge FG\neg p \rightarrow F(HFp \wedge G\neg p)$		
	(b)	$Pp \wedge PH\neg p \rightarrow P(GPp \wedge H\neg p)$		
(A8)		$H(Hp \rightarrow p) \rightarrow Hp$		
(A9)	(a)	$FGp \rightarrow GFp$	(b)	$PHp \rightarrow HPp$.

A few definitions are needed before we can state precisely the basic problem of tense logic, that of finding characteristic axioms that 'correspond' to various assumptions about Time.

1.4 Formal Semantics

A *frame* is a nonempty set C equipped with a binary relation R. A *valuation* in a frame (X, R) is a function V assigning each variable p_i a subset of X. Intuitively, X can be thought of as representing the set of instants of time, R

the earlier-later relation, V the function telling us *when* each p_i is the case. We extend V to a function defined on *all* formulas, by abuse of notation still called V, inductively as follows:

$$\begin{array}{lcl} V(\neg\alpha) & = & X - V(\alpha) \\ V(\alpha \wedge \beta) & = & V(\alpha) \cap V(\beta) \\ V(G\alpha) & = & \{x \in X : \forall y \in X(xRy \rightarrow y \in V(\alpha))\} \\ V(H\alpha) & = & \{x \in X : \forall y \in X(yRx \rightarrow y \in V(\alpha))\}. \end{array}$$

(Some writers prefer a different notion. Thus, what we have expressed as $x \in V(\alpha)$ may appear as $\|\alpha\|_x^V = \mathrm{TRUE}$ or as $(X, R, V) \vDash \alpha[x]$.) A formula α is *valid* in a frame (X, R) if $V(\alpha) = X$ for every valuation V in (X, R), and is *satisfiable* in (X, R) if $V(\alpha \neq \varnothing$ for some valuation V in (X, R), or equivalently if $\neg\alpha$ is not valid in (X, R). Further, α is *valid* over a class $\mathcal{K}$ of frames if it is valid in every $(X, R) \in \mathcal{K}$, and is *satisfiable* over $\mathcal{K}$ if it is satisfiable in some $(X, R) \in \mathcal{K}$, or equivalently if $\neg\alpha$ is not valid over $\mathcal{K}$. A system $\mathbf{L}$ in standard format is *sound* for $\mathcal{K}$ if every thesis of $\mathbf{L}$ is valid over $\mathcal{K}$, and a sound system $\mathbf{L}$ is *complete* for $\mathcal{K}$ if conversely every formula valid over $\mathcal{K}$ is a thesis of $\mathbf{L}$, or equivalently, if every formula consistent with $\mathbf{L}$ is satisfiable over $\mathcal{K}$. Any set (let us say, finite) Φ of first- or second-order axioms about the earlier-later relation $<$ determines a class $\mathcal{K}(\Phi)$ of frames, the class of its *models.* The basic correspondence problem of tense logic is, given Φ to find characteristic axioms for a system $\mathbf{L}$ that will be sound and complete for $\mathcal{K}(\Phi)$. The next two sections of this survey will be devoted to representing the solution to this problem for many important Φ.

1.5 Motivation

But first it may be well to ask, why bother? Several classes of motives for developing an autonomous tense logic may be cited:

(a) *Philosophical* motives were behind much of the pioneering work of A. N. Prior, to whom the following point seemed most important: whereas our ordinary language is tensed, the language of physics is mathematical and so untensed. Thus, there arise opportunities for confusions between different 'terms of ideas'. Now working in tense logic, what we learn is precisely how to avoid confusing the tensed and the tenseless, and how t clarify their relations (e.g. we learn that essentially the same thought can be formulated tenselessly as, 'Of any two distinct instants, one /is/ earlier and the other /is/ later', and tensedly as, 'Whatever is going to have been the case either already has been or now is or is sometime going to be the case). Thus, the study of tense logic can have at least a 'therapeutic' value. Later writers have stressed other philosophical applications, and some of these are treated elsewhere in this *Handbook.*

(b) *Exegetical* applications again interested Prior (see his [Prior, 1967, Chapter 7]). Much was written about the logic of time (especially about future contingents) by such ancient writers as Aristotle and Diodoros Kronos (whose works are unfortunately lost) and by such mediaeval ones as William of Ockham or Peter Auriole. It is tempting to try to bring to bear insights from modern logic to the interpretation of their thought. But to pepper the text of an Aristotle or an Ockham with such regimenters' phrases as 'at time t' is an almost certain guarantee of misunderstanding. For these earlier writers thought of such an item as 'Socrates is running' as being already complete as it stands, *not* as requiring supplementation before it could express a proposition or have a truth-value. Their standpoint, in other words, was like that of modern tense logic, whose notions and notations are likely to be of most use in interpreting their work, if any modern developments are.

(c) *Linguistic* motivations are behind much recent work in tense logic. A certain amount of controversy surrounds the application of tense logic to natural language. See, e.g. van Benthem [1978; 1981] for a critic's views. To avoid pointless disputes it should be emphasised from the beginning that tense logic does not attempt the faithful replication of every feature of the deep semantic structure (and still less of the surface syntax) of English or any other language; rather, it provides an idealised model giving the sympathetic linguist food for thought. an example: in tense logic, P and F can be iterated indefinitely to form, e.g. $PPPFp$ or $FPFPp$. In English, there are four types of verbal modifications indicating temporal reference, each applicable at most *once* to the main verb of a sentence: Progressive (be + ing), Perfect (have + en), Past (+ ed), and Modal auxiliaries (including will, would). Tense logic, by allowing unlimited iteration of its operators, departs from English, to be sure. But by doing so, it enables us to raise the question of whether the multiple compounds formable by such iteration are really all distinct in meaning; and a theorem of tense logic (see Section 3.5 below) tells us that on reasonable assumptions they are not, e.g. $PPPFp$ and $FPFPp$ both collapse to PFp (which is equivalent to PPp). and this may suggest *why* English does not *need* to allow unlimited iteration of its temporal verb modifications.

(d) *Computer Science*: Both tense logic itself and, even more so, the closely related so-called *dynamic logic* have recently been the objects of much investigation by theorists interested in program verification. temporal operators have been used to express such properties of programs as termination, correctness, safety, deadlock freedom, clean behaviour, data integrity, accessibility, responsiveness, and fair scheduling. These studies are mainly concerned only with *future* temporal operators, and so fall technically within the province of *modal* logic. See Harel *et al.*'s chapter on dynamic logic in Volume 4 of this *Handbook*, Pratt [1980] among other items in our bibliography.

(e) *Mathematics*: Some taste of the purely mathematical interest of tense logic will, it is hoped, be apparent from the survey to follow. Moreover, tense logic is not an isolated subject within logic, but rather has important links with modal logic, intuitionistic logic, and (monadic) second-order logic.

Thus, the motives for investigating tense logic are many and varied.

2 FIRST STEPS IN TENSE LOGIC

Let $\mathbf{L}_0$ be the system in standard format with characteristic axioms (A0a, b, c, d). Let $\mathcal{K}_0$ be the class of *all* frames. We will show that $\mathbf{L}_0$ is (sound and) complete for $\mathcal{L}_0$, and thus deserves the title of *minimal* tense logic. The method of proof will be applied to other systems in the next section. Throughout this section, thesishood and consistency are understood relative to $\mathbf{L}_0$, validity and satisfiability relative to $\mathcal{K}_0$.

THEOREM 1 (Soundness Theorem). $\mathbf{L}_0$ *is sound for* $\mathcal{K}_0$.

Proof. We must show that any thesis (of $\mathbf{L}_0$) is valid (over $\mathcal{K}_0$). for this it suffices to show that each axiom is valid, and that each rule preserves validity. the verification that tautologies are valid, and that substitution and MP preserves validity is a bit tedious, but entirely routine.

To check that $(A0a)$ is valid, we must show that for all relevant X, R, V and x, if $x \in V(G(p \to q0)$ and $x \in V(Gp)$, then $x \in V(Gq)$. Well, the hypotheses here mean, first that whenever xRy and $y \in V(p)$, then $y \in V(q)$; and second that whenever xRy, then $y \in V(p)$. The desired conclusion is that whenever xRy, then $y \in V(q)$; which follows immediately. Intuitively, (A0a) says that if q is going to be the case whenever p is, and p is always going to be the case, then q is always going to be the case. The treatment of (A0b) is similar.

To check that (A0c) is valid, we must show that for all relevant X, R, V, and x, if $x \in V(p)$, then $x \in V(GPp)$. Well, the desired conclusion here is that for every y with xRy there is a z with zRy and $z \in V(p)$. It suffices to take $z = x$. Intuitively, (A0c) says that whatever is now the case is always going to have been the case. The treatment of (A0d) is similar.

To check that TG preserves validity, we must show that if for all relevant X, R, V, and x we have $x \in V(\alpha)$, then for all relevant X, R, V, and x we have $x \in V(H\alpha)$ and $x \in V(G\alpha)$, in other words, that whenever yRx we have $y \in V(\alpha)$ and whenever xRy we have $y \in V(\alpha)$. But this is immediate. Intuitively, TG says that if something is now the case *for logical reasons alone*, then for logical reasons alone it always has been and is always going to be the case: logical truth is eternal. ■

In future, verifications of soundness will be left as exercises for the reader. Our proof of the completeness of $\mathbf{L}_0$ for $\mathcal{K}_0$ will use the method of maximal consistent sets, first developed for first-order logic by L. Henkin, systematically applied to tense logic by E. J. Lemmon and D. Scott (in notes eventually published as [Lemmon and Scott, 1977]), and refined [Gabbay, 1975].

The completeness of $\mathbf{L}_0$ for $\mathcal{K}_0$ is due to Lemmon. We need a number of preliminaries.

THEOREM 2 (Derived rules). *The following rules of inference preserve thesishood:*

1. *from* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *to infer any truth- functional consequence* β
2. *from* $\alpha \to \beta$ *to infer* $G\alpha \to G\beta$ *and* $H\alpha \to H\beta$
3. *from* $\alpha \leftrightarrow \beta$ *and* $\theta(\alpha/p)$ *to infer* $\theta(\beta/p)$
4. *from* α *to infer its mirror image.*

Proof.

1. To say that β is a truth-functional consequence of $\alpha_1, \alpha_2, \ldots, \alpha_n$ is to say that $(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n \to \beta)$ or equivalently $\alpha_1 \to (\alpha_2 \to (\ldots(\alpha_n \to \beta)\ldots))$ is an instance of a tautology, and hence is an axiom. We then apply MP.
2. From $\alpha \to \beta$ we first obtain $G(\alpha \to \beta)$ by TG, and then $G\alpha \to G\beta$ by A0a and MP. Similarly for H.
3. Here (α/p) denotes substitution of α for the variable p. It suffices to prove that if $\alpha \to \beta$ and $\beta \to \alpha$ are theses, then so are $\theta(\alpha/p) \to \theta(\beta/p)$ and $\theta(\beta/p) \to \theta(\alpha/p)$. This is proved by induction on the complexity of θ, using part (2) for the cases $\theta = G\chi$ and $\theta = H\chi$. In particular, part (3) allows us to insert and remove double negations freely. We write $\alpha \approx \beta$ to indicate that $\alpha \leftrightarrow \beta$ is a thesis.
4. This follows from the fact that the tense-logical axioms of $\mathbf{L}_0$ come in mirror-image pairs, (A0a, b) and (A0c, d). Unlike parts (1)–(3), part (4) will *not* necessarily hold for every extension of $\mathbf{L}_0$. ∎

THEOREM 3 (Theses). *Items (a)–(h) below are theses of* $\mathbf{L}_0$.

Proof. We present a deduction, labelling some of the lines as theses for future reference:

	(1)	$G(p \to q) \to G(\neg q \to \neg p)$	from a tautology by 1.2b
	(2)	$G(\neg q \to \neg p) \to (G\neg q \to G\neg p)$	(A0a)
(a)	(3)	$G(p \to q) \to (Fp \to Fq)$	from 1,2 by 1.2a
	(4)	$Gp \to G(q \to p \wedge q)$	from a tautology by 1.2b
	(5)	$G(q \to p \wedge q) \to (Fq \to F(p \wedge q))$	3
(b)	(6)	$Gp \wedge Fq \to F(p \wedge q)$	from 4, 5 by 1.2a
	(7)	$p \to GPp$	(A0c)
	(8)	$GPp \wedge Fq \to F(Pp \wedge q)$	6
(c)	(9)	$p \wedge Fq \to F(Pp \wedge q)$	from 7, 8 by 1.2a
	(10)	$G(p \wedge q) \to Gp$	
		$G(p \wedge q) \to Gq$	from tautologies by 1.2b
	(11)	$G(q \to p \wedge q) \to (Gq \to G(p \wedge q))$	(A0a)
(d)	(12)	$Gp \wedge Gq \leftrightarrow G(p \wedge q)$	12
	(14)	$G\neg p \wedge G\neg q \to G\neg(p \vee q)$	from 13 by 1.3c
(e)	(15)	$Fp \vee Fq \leftrightarrow F(p \vee q)$	from 14 by 1.2a
	(16)	$Gp \to G(p \vee q)$	
		$Gq \to G(p \vee q)$	from tautologies by 1.2b
(f)	(17)	$Gp \vee Gq \to G(p \vee q)$	from 16 by 1.2a
	(18)	$G\neg q \vee G\neg q \to G(\neg p \vee \neg q)$	17
	(19)	$G\neg p \vee G\neg q \to G\neg(p \wedge q)$	from 18 by 1.2c
(g)	(20)	$F(p \wedge q \to Fp \wedge Fq$	from 19 by 1.2a
	(21)	$\neg p \to HF\neg p$	(A0d)
	(22)	$\neg p \to H\neg Gp$	from 21 by 1.2c
(h)	(23)	$PGp \to p$	from 22 by 1.2a

Also the mirror images of 1.3a–h are theses by 1.2d. ■

We assume familiarity with the following:

LEMMA 4 (Lindenbaum's Lemma). *Any consistent set of formulas can be extended to a maximal consistent set.*

LEMMA 5. *Let Q be a maximal consistent set of formulas. For all formulas we have:*

1. *If $\alpha_1, \ldots, \alpha_n \in A$ and $\alpha_1 \wedge \ldots \wedge \alpha_n \to \beta$ is a thesis, then $\beta \in A$.*
2. *$\neg\alpha \in A$ iff $\alpha \notin A$*
3. *$(\alpha \wedge \beta) \in A$ iff $\alpha \in A$ and $\beta \in A$*
4. *$(\alpha \vee \beta) \in A$ iff $\alpha \in A$ or $\beta \in A$.*

They will be used tacitly below.

Intuitively, a maximal consistent set—henceforth abbreviated MCS—represents a full description of a possible state of affairs. For MCSs A, B we say that A is *potentially followed* by B, and write $A \mathrel{-\!\!\!3} B$, if the conditions of Lemma 6 below are met. Intuitively, this means that a situation of the sort described by A could be followed by one of the sort described by B.

LEMMA 6. *For any* MCS*s* A, B*, the following are equivalent:*

1. *whenever* $\alpha \in A$*, we have* $P\alpha \in B$*,*
2. *whenever* $\beta \in B$*, we have* $F\beta \in A$*,*
3. *whenever* $G\gamma \in A$*, we have* $\gamma \in B$*,*
4. *whenever* $H\delta \in B$*, we have* $\delta \in A$*.*

Proof. To show (1) implies (3): assume(1) and let $G\gamma \in A$. Then $PG\gamma \in B$, so by Thesis 3(h) we have $\gamma \in B$ as required by (3).

To show (3) implies (2): assume (3) and let $\beta \in B$. then $\neg\beta \notin B$, so $G\neg\beta \notin A$, and $F\beta = \neg G\neg\beta \in A$ as required by (2).

Similarly (2) implies (4) and (4) implies (1). ■

LEMMA 7. *Let* C *be an* MCS*,* γ *any formula:*

1. *if* $F\gamma \in C$*, then there exists an* MCS B *with* $C \mathrel{-\!\!\!3} B$ *and* $\gamma \in B$*,*
2. *if* $P\gamma \in C$*, then there exists an* MCS A *with* $A \mathrel{-\!\!\!3} C$ *and* $\gamma \in A$*.*

Proof. We treat (1): it suffices (by the criterion of Lemma 6(a)) to obtain an MCS B containing $B_0 = \{P\alpha : \alpha \in C\} \cup \{\gamma\}$. For this it suffices (by Lindenbaum's Lemma) to show that B_0 is consistent. For this it suffices (by the closure of C under conjunction plus the mirror image of Theorem 3(g)) to show that for any $\alpha \in C, P\alpha \wedge \gamma$ is consistent. For this it suffices (since TG guarantees that $\neg F\delta$ is a thesis whenever $\neg\delta$ is) to show that $F(P\alpha \wedge \gamma)$ is consistent. And for this it suffices to show that $F(P\alpha \wedge \gamma)$ belongs to C—as it must by 3(c). ■

DEFINITION 8. A *chronicle* on a frame (X, R) is a function T assigning each $x \in X$ an MCS $T(x)$. Intuitively, if X is thought of as representing the set of instants, and R the earlier-later relation, T should be thought of as providing a complete description of what goes on at each instant. T is *coherent* if we have $T(x) \mathrel{-\!\!\!3} T(y)$ whenever xRy. T is *prophetic* (resp. *historic*) if it is coherent and satisfies the first (resp. second) condition below:

1. whenever $F\gamma \in T(x)$ there is a y with xRy and $\gamma \in T(y)$,
2. whenever $P\gamma \in T)(x)$ there is a y with yRx and $\gamma \in T(y)$.

T is *perfect* if it is both prophetic and historic. Note that T is coherent iff it satisfies the two following conditions:

3. whenever $\gamma \in T(x)$ and xRy, then $\gamma \in T(y)$,

4. whenever $H\gamma \in T(x)$, and yRx, then $\gamma \in T(y)$.

If V is a valuation in (X, R), the *induced* chronicle T_V is defined by $T_V(x) = \{\gamma : x \in V(\gamma_0\}; T_V$ is always perfect. If T is a perfect chronicle on (X, R), the *induced* valuation V_T is defined by $V_T(p_i) = \{x : p_i \in T(x)\}$. We have:

LEMMA 9 (Chronicle Lemma). *Let T be a perfect chronicle on a frame (X, R). If $V = V_T$ is the valuation induced by T, then $T = T_V$ the chronicle induced by V. In other words, for all formulas γ we have:*

$$(+) \qquad V(\gamma) = \{x : \gamma \in T(x)\}$$

In particular, any member of any $T(x)$ is satisfiable in (X, R).

Proof. (+) is proved by induction on the complexity of γ. As a sample, we treat the induction step for G: assume (+) for γ, to prove it for $G\gamma$:

On the one hand, if $G\gamma \in T(x)$, then by Definition 8(3), whenever xRy we have $\gamma \in T(y)$ and by induction hypothesis $y \in V(\gamma)$. This shows $x \in V(G\gamma)$.

On the other hand, if $G\gamma \notin T(x)$, then $F\neg\gamma \approx \neg G\gamma \in T(x)$, so by Definition 8(1) for some y with xRy we have $\neg\gamma \in T(y)$ and $\gamma \notin T(y)$, whence by induction hypothesis, $y \notin V(\gamma)$. This shows $x \notin V(G\gamma)$. ■

To prove the completeness of $\mathbf{L}_0$ for $\mathcal{K}_0$ we must show that every consistent formula γ_0 is satisfiable. Now Lemma 9 suggests an obvious strategy for proving γ_0 satisfiable, namely to construct a perfect chronicle T on some frame (X, R) containing an x_0 with $\gamma_0 \in T(x_0)$. We will construct X, R, and T piecemeal.

DEFINITION 10. Fix a denumerably infinite set W. Let M be the set of all triples (X, R, T) such that :

1. X is a nonempty finite subset of W,

2. R is an antisymmetric binary relation on X,

3. T is a coherent chronicle on (X, R).

For $\mu = (X, R, T)$ and $\mu' = (X', R', T')$ in M we say μ' *extends* μ if (when relations and functions are identified with sets f ordered pairs) we have:

$1'$. $\quad X \subseteq X'$

$2'$. $\quad R = R' \cap (X \times X)$

$3'$. $\quad T \subseteq T'$.

A conditional requirement of form 8(1) or (2) will be called *unborn* for $\mu = (X, R, T) \in M$ if its antecedent is not fulfilled; that is, if $x \notin X$ or if $x \in X$ but $F\gamma$ or $P\gamma$ a the case may be does not belong to $T(x)$. It will be called *alive* for μ if its antecedent is fulfilled but its consequent is not; in other words, there is no $y \in X$ with xRy or yRx as the case may be and $\gamma \in T(y)$. It will be called *dead* for μ if its consequent is fulfilled.

Perhaps no member of M is perfect; but any imperfect member of M can be improved:

LEMMA 11 (Killing Lemma). *Let $\mu = (X, R, T) \in M$. For any requirement of form 8(1) or (2) which is alive for μ, there exists an extension $u' = (X', R', T') \in M$ of μ for which that requirement is dead.*

Proof. We treat a requirement of form 8(1). If $x \in X$ and $F\gamma \in T(x)$, by 7(1) there is an MCS B with $T(x) \multimap\!\!\!\!\exists\, B$ and $\gamma \in B$. It therefore suffices to fix $y \in W - X$ and set

1. $X' = X \cup \{y\}$
2. $R' = R \cup \{(x, y)\}$
3. $T' = T \cup \{(y, B)\}$. ■

THEOREM 12 (Completeness Theorem). $\mathbf{L}_0$ *is complete for* $\mathcal{K}_0$.

Proof. Given a consistent formula γ_0, we wish to construct a frame (X, R) and a perfect chronicle T on it, with $\gamma_0 \in t(x_0)$ for some x_0. To this end we fix an enumeration $x_0, x_1, x_2, \ldots$ of W, and an enumeration $\gamma_0, \gamma_1, \gamma_2, \ldots$ of all formulas. To the requirement of form 8(1) (resp. 8(2)) for $x = x_i$ and $\gamma = \gamma_j$ we assign the *code number* $2 \cdot 5^{i} \cdot 7^{j}$ (resp. $3 \cdot 5^{i} \cdot 7^{j}$). Fix an MCS C_0 with $\gamma_0 \in C_0$, and let $\mu_0 = (X_0, R_0, T_0)$ where $X_0 = \{x_0\}, R_0 = \varnothing$, and $T_0 = \{(x_0, C_0)\}$. If μ_n is defined, consider the requirement, which among all those which are alive for μ_n, has the least code number. Let μ_{n+1} be an extension of μ_n for which that requirement is dead, as provided by the Killing Lemma. Let (X, R, T) be the union of the $\mu_n = (X_n, R_n, T_n)$; more precisely, let X be the union of the X_n, R of the R_n, and T of the T_n. It is readily verified that T is a perfect chronicle on (X, R), as required. ■

The observant reader may be wondering why in Definition 10(2) the relation R was required to be antisymmetric. the reason was to enable us to make the following remark: our proof actually shows that every thesis of $\mathbf{L}_0$ is valid over the class $\mathcal{K}_0$ of all frames, and that every formula consistent with $\mathbf{L}_0$ is satisfiable over the class $\mathcal{K}_{\text{anti}}$ of antisymmetric frames. Thus, $\mathcal{K}_0$ and $\mathcal{K}_{\text{anti}}$ give rise to the same tense logic; or to put the matter differently, there is no characteristic axiom for tense logic which 'corresponds' to the assumption that the earlier-later relation on instants of time is antisymmetric.

In this connection a remark is in order: suppose we let X be the set of *all* MCSs, R the relation $\dashv\!\exists$, V the valuation $V(p_i) = \{x : p_i \in x\}$. Then using Lemmas 6 and 7 it can be checked that $V(\gamma) = \{x : \gamma \in x\}$ for *all* γ. In this way we get a quick proof of the completeness of $\mathbf{L}_0$ for $\mathcal{K}_0$. However, this (X, R) is not antisymmetric. Two MCSs A and B may be *clustered* in the sense that $A \dashv\!\exists B$ and $B \dashv\!\exists A$. There is a trick, known as 'bulldozing', though, for converting nonantisymmetric frames to antisymmetric ones, which can be used here to give an alternative proof of the completeness of $\mathbf{L}_0$ for $\mathcal{K}_{\mathrm{anti}}$. See Bull and Segerberg's chapter in Volume 3 of this *Handbook* and [Segerberg, 1970].

3 A QUICK TRIP THROUGH TENSE LOGIC

The material to be presented in this section was developed piecemeal in the late 1960s. In addition to persons already mentioned, R. Bull, N. Cocchiarella and S. Kripke should be cited as important contributors to this development. Since little was published at the time, it is now hard to assign credits.

3.1 Partial Orders

Let $\mathbf{L}_1$ be the extension for $\mathbf{L}_0$ obtained by adding (A1a) as an extra axiom. Let $\mathcal{K}_1$ be the class of *partial orders*, that is, of antisymmetric, transitive frames. We claim $\mathbf{L}_1$ is (sound and) complete for $\mathcal{K}_1$. Leaving the verification of soundness as an exercise for the reader, we sketch the modifications in the work of the preceding section needed to establish completeness.

First of all, we must now understand the notions of thesishood and consistency and, hence, of MCS and chronicle, as relative to $\mathbf{L}_0$. Next, we must revise clause 10(2) in the definition of M to read:

2_1. R is a partial order on X.

This necessitates a revision in clause 11(2) in the proof of the Killing Lemma. Namely, in order to guarantee that R' will be a partial order on X', that clause must now read:

2_1. $R' = R \cup \{(x, y)\} \cup \{(v, y) : vRx\}$.

But now it must be checked that T', as defined by clause 11(3), remains a coherent chronicle under the revised definition of R'. Namely, it must be checked that if vRx, then $T(v) \dashv\!\exists B$. To show this (and so complete the proof) the following suffices:

LEMMA *Let* A, C, B *be* MCS*s. If* $A \dashv\!\exists C$ *and* $C \dashv\!\exists B$*, then* $A \dashv\!\exists B$*.*

Proof. We use criterion 6(3) for $\dashv\!\exists$: assume $G\gamma \in A$, to prove $\gamma \in B$. Well, by the new axiom (A1a) we have $GG\gamma \in A$. Then since $A \dashv\!\exists\, C$, we have $G\gamma \in C$, and since $C \dashv\!\exists\, B$, we have $\gamma \in B$. ■

It is worth remarking that the mirror image (A1b) of (A1a) is equally valid over partial orders, and must thus by the completeness theorem be a thesis of $\mathbf{L}_0$. To find a deduction of it is a nontrivial exercise.

3.2 Total Orders

Let $\mathbf{L}_2$ be the extension of $\mathbf{L}_1$ obtained by adding (A2a, b) as extra axioms. Let $\mathcal{K}_1$ be the class of *total orders*, or frames satisfying antisymmetry, transitivity, and comparability. Leaving the verification of soundness to the reader, we sketch the modifications in the work of Section 3.1 above, beyond simply understanding thesishood and related notions as relative to $\mathbf{L}_2$, needed to show $\mathbf{L}_2$ complete for $\mathcal{K}_2$.

To begin with, we must revise clause 10(2) in the definition of M to read:

2_2. R is a partial order on X.

This necessitates revisions in the proof of the Killing Lemma, for which the following will be useful:

LEMMA *Let* A, B, C *be* MCSs*. If* $A \dashv\!\exists\, B$ *and* $A \dashv\!\exists\, C$*, then either* $B = C$ *or* $B \dashv\!\exists\, C$ *or* $C \dashv\!\exists\, B$.

Proof. Suppose for contradiction that the two hypotheses hold but none of the three alternatives in the conclusion holds. Using criterion 6(2) for $\dashv\!\exists$, we see that there must exist a $\gamma_0 \in C$ with $F\gamma_0 \notin b$ (else $B \dashv\!\exists\, C$) and a $\beta_0 \in B$ with $F\beta_0 \notin C$ (else $C \dashv\!\exists\, B$). Also there must exist a δ with $\delta \in B, \delta \notin C$ (else $B = C$). Let $\beta = \beta_0 \wedge \neg F\gamma_0 \wedge \delta \in B, \gamma = \gamma_0 \wedge \neg F\beta_0 \wedge \neg\delta \in C$. We have $F\beta \in A$ (since $A \dashv\!\exists\, B$) and $F\gamma \in A$ (since $A \dashv\!\exists\, C$). hence, by A2a, one of $F(\beta \wedge F\gamma), F(F\beta \wedge \gamma), F(\beta \wedge \gamma)$ must belong to A. But this is impossible since all three are easily seen (using 3(7)) to be inconsistent. ■

Turning now to the Killing Lemma, consider a requirement of form 8(1) which is alive for a certain $\mu = (X, R, T) \in M$. We claim there is an extension $\mu' = (X', R', T')$ for which it is dead. This is proved by induction on the number n of successors which x has in (X, R). We fix an MCS B with $T(x) \dashv\!\exists\, B$ and $\gamma \in B$. If $n = 0$, it suffices to define μ' as was done in Section 3.1 above.

If $n > 0$, let x' be the immediate successor of x in (X, R). We cannot have $\gamma \in T(x')$ or else our requirement would already be dead for μ. If $F\gamma \in T(x')$, we can reduce to the case $n - 1$ by replacing x by x'. So suppose $F\gamma \notin T(x')$. Then we have neither $B = T(x')$ nor $T(x') \dashv\!\exists\, B$.

Hence, by the Lemma, we must have $B {-\!\!\exists}\, T(x')$. Therefore it suffices to fix $y \in W - X$ and set:

$$\begin{array}{lcl} X' & = & X \cup \{y\} \\ R' & = & R]cup\{(x,y),(y,x')\} \cup \{(v,y) : vRx\} \cup \{(y,v) : (x'Rv)\} \\ I' & = & T \cup \{(y,B)\}. \end{array}$$

In other words, we insert a point between x and x', assigning it the set B. Requirements of form 8(2) are handled similarly, using a mirror image of the Lemma, proved using (A2b). No further modifications in the work of Section 3.1 above are called for.

The foregoing argument also establishes the following: let $\mathbf{L}_{\text{tree}}$ be the extension of $\mathbf{L}_1$ obtained by adding (A2b) as an extra axiom. Let $\mathcal{K}_{\text{tree}}$ be the class of *trees*, defined for present purposes as those partial orders in which the predecessors of any element are totally ordered. Then $\mathbf{L}_{\text{tree}}$ is complete for $\mathcal{K}_{\text{tree}}$.

It is worth remarking that the following are valid over total orders:

$$FPp \to Pp \vee p \vee Fp, \quad PFp \to Pp \vee p \vee Fp.$$

To find deductions of them in $\mathbf{L}_2$ is a nontrivial exercise. As a matter of fact, these two items could have been used instead of (A2a, b) as axioms for total orders. One could equally well have used their contrapositives:

$$Hp \wedge p \wedge Gp \to GHp, \quad Hp \wedge p \wedge Gp \to HGp.$$

The converses of these four items are valid over partial orders.

3.3 No Extremals (No Maximals, No Minimals)

Let $\mathbf{L}_3$ (resp. $\mathbf{L}_4$) be the extension of $\mathbf{L}_2$ obtained by adding (A3a, b) (resp. (A4a, b)) as extra axioms. Let $\mathcal{K}_3$ (resp. $\mathcal{K}_4$) be the class of total orders having (resp. not having) a maximum and a minimum. Beyond understanding the notions of consistency and MCS relative to $\mathbf{L}_3$ or $\mathbf{L}_4$ as the case may be, no modification in the work of Section 3.2 above is needed to prove $\mathbf{L}_3$ complete for $\mathcal{K}_3$ and $\mathbf{L}_4$ for $\mathcal{K}_4$. The following observations suffice:

On the one hand, understanding consistency and MCS relative to $\mathbf{L}_3$, if (X, R) is any total order and T any perfect chronicle on it, then for any $x \in X$, either $G\bot \in T(x)$ itself, or $FG\bot \in T(x)$ and so $G\bot \in t(y)$ for some y with xRy—this by (A3a). But if $G\bot \in T(z)$, then with w with zRw would have to have $\bot \in T(w)$, which is impossible so z must be the maximum of (X, R). Similarly, A3b guarantees the existence of a minimum in (X, R).

On the other hand, understanding consistency and MCS relative to $\mathbf{L}_4$, if (X, R) is any total order and T any perfect chronicle on it, then for any

$x \in X$ we have $G\top \to F\top \in T(x)$, and hence $F\top \in T(x)$, so there must be a y with ($\top \in T(y)$ and) xRy— this by (A4a). Similarly, (A4b) guarantees that for any x there is a y with yRx.

The foregoing argument also establishes that the extension of $\mathbf{L}_1$ obtained by adding (A4a, b) is complete for the class of partial orders having nonmaximal or minimal elements.

It hardly needs saying that one can axiomatise the view (characteristic of Western religious cosmologies) that Time had a beginning, but will have no end, by adding (A3b) and (A4a) to $\mathbf{L}_2$.

3.4 Density

The extension $\mathbf{L}_5$ of $\mathbf{L}_2$ obtained by adding (A5a) (or equivalently (A5b)) is complete for the class $\mathcal{K}_5$ of dense total orders. The main modification in the work of Section 3.2 above needed to show this is that in addition to requirements of forms 8(1,2) we need to consider requirements of the form:

5. if xRy, then there exists a z with xRz and zRy.

To 'kill' such a requirement, given a coherent chronicle T on a finite total order $(X, R$ and $x, y \in X$ with y immediately succeeding x, we need to be able to insert a point z between x and y, and find a suitable MCS to assign to z. For this the following suffices:

LEMMA *Let* A, B *be* MCS*s with* $A \dashv B$. *Then there exists an* MCS C *with* $A \dashv C$ *and* $C \dashv B$.

Proof. The problem quickly reduces to showing $\{P\alpha : \alpha \in A\} \cup \{F\beta : \beta \in B\}$ consistent. For this it suffices to show that if $\alpha \in A$ and $\beta \in b$, then $F(P\alpha \wedge F\beta) \in A$. Now if $\beta \in B$, then since $A \dashv B, F\beta \in A$, and by (A5a), $FF\beta \in A$. An appeal to 3(3) completes the proof. ■

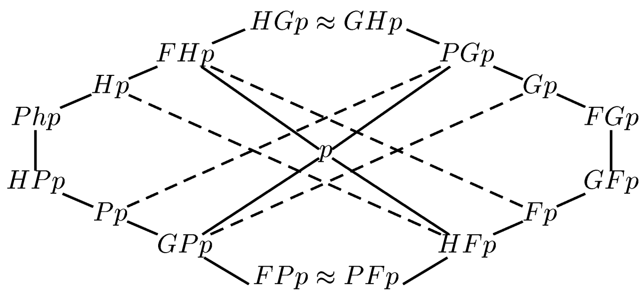

Figure 1.

Table 1.

$GGHp \approx GHp$	$FGHp \approx GHp$
$GFHp \approx GHp$	$FFHp \approx FHp$
$GPGp \approx Gp$	$FPGp \approx FGp$
$GPHp \approx PHp$	$FPHp \approx PHp$
$GFGp \approx FGp$	$FFGp \approx FGp$
$GHPp \approx HPp$	$FHPp \approx HPp$
$GGFp \approx GFp$	$FGFp \approx GFp$
$GGPp \approx GPp$	$FGPp \approx FPp$
$GHFp \approx GFp$	$FHFp \approx Fp$
$GFPp \approx FPp$	$FFPp \approx FPp$

Similarly, the extension $\mathbf{L_Q}$ of $\mathbf{L}_2$ obtained by adding (A4a, b) and (A5a) is complete for the class of dense total orders without maximum or minimum. A famous theorem tells us that any *countable* order of this class is isomorphic to the rational numbers in their usual order. Since our method of proof always produces a *countable* frame, we can conclude that $\mathbf{L_Q}$ is the tense logic of the rationals. The accompanying diagram (1) indicates some implications that are valid over dense total orders without maximum or minimum, and hence theses of $\mathbf{L_Q}$; no further implications among the formulas considered are valid. A theorem of C. L. Hamblin tells us that in $\mathbf{L_Q}$ any sequence of Gs, Hs, Fs and Ps prefixed to the variable p is provably equivalent to one of the 15 formulas in our diagram. It obviously suffices to prove this for sequences of length three. The reductions listed in the accompanying Table 1 together with their mirror images, suffice to prove this. It is a pleasant exercise to verify all the details.

3.5 Discreteness

The extension $\mathbf{L}_6$ of $\mathbf{L}_2$ obtained by adding (A6a, b) is complete for the class $\mathcal{K}_6$ of total orders in which every element has an immediate successor and an immediate predecessor. The proof involves quite a few modifications in the work of Section 3.2 above, beginning with:

LEMMA *For any* MCS A *there exists an* MCS B *such that:*

1. *whenever* $F\gamma \in A$ *then* $\gamma \vee F\gamma \in B$.

Moreover, any such MCS *further satisfies:*

2. *whenever* $P\delta \in B$, *then* $\delta \vee P\delta \in A$,

3. *whenever* $A \mathrel{-\!\!\exists} C$*, then either* $B = C$ *or* $B \mathrel{-\!\!\exists} C$*,*

4. *whenever* $C \mathrel{-\!\!\exists} B$*, then either* $A = C$ *or* $C \mathrel{-\!\!\exists} A$*.*

Proof.

1. The problem quickly reduces to proving the consistency of any finite set of formulas of the forms $P\alpha$ for $\alpha \in A$ and $\gamma \vee F\gamma$ for $F\gamma \in A$. To establish this, one notes that the following is valid over total orders, hence a thesis of ($\mathbf{L}_2$ and *a fortiori* of) $\mathbf{L}_6$:

$$Fp_0 \wedge Fp_1 \wedge \ldots \wedge Fp_n \rightarrow$$
$$F((p_0 \vee Fp_0) \wedge (p_1 \vee Fp_1) \wedge \ldots \wedge (p_n \vee Fp_n))$$

2. We prove the contrapositive. Suppose $\delta \vee P\delta \notin A$. By (A6a), $FH\neg\delta \in A$. by part (1), $H\neg\delta \vee FH\neg\delta \in B$. But $FHp \rightarrow Hp$ is valid over total orders, hence a thesis of $\mathbf{L}_2$ and *a fortiori* of) $\mathbf{L}_6$. So $H\neg\delta \in B$ and $P\delta \notin B$ as required.

3. Assume for contradiction that $A \mathrel{-\!\!\exists} C$ but neither $B = C$ nor $B \mathrel{-\!\!\exists} C$. Then there exist a $\gamma_0 \in C$ with $\gamma_0 \notin B$ and a $\gamma_1 \in C$ with $F\gamma_1 \notin B$. Let $\gamma = \gamma_0 \wedge \gamma_1$. Then $\gamma \in C$ and since $A \mathrel{-\!\!\exists} C, F\gamma \in A$. but $\gamma \vee F\gamma \notin B$, contrary to (1).

4. Similarly follows from (2). ■

We write $A \mathrel{-\!\!\exists}' B$ to indicate that A, B are related as in the above Lemma. Intuitively this means that a situation of the sort described by A could be *immediately* followed by one of the sort described by B.

We now take M to e the set of *quadruples* (X, R, S, T) where on the one hand, as always X is a nonempty finite subset of W, R a total order on X, and T a coherent chronicle on (X, R); while on the other hand, we have:

4. whenever xSy, then y immediately succeeds x in (X, R),

5. whenever xSy, then $T(x) \mathrel{-\!\!\exists}' T(y)$,

Intuitively xSy means that no points are ever to be added between x and y. We say (X', R', S', T') *extends* (X, R, S, T) if on the one hand, as always, Definition 10(1′, 2′, 3′) hold; while on the other hand, $S \subseteq S'$. In addition to requirements of the form 8(1, 2) we need to consider requirements of the form:

5. there exists a y with xSy,

4. there exists a y with ySx.

To 'kill' a requirement of form (5), take an MCS B with $T(x) \mathrel{-\!\!3}' B$. If x is the maximum of (X, R) it suffices to fix $z \in W - X$ and set:

$$\begin{array}{lllllll} X' & = & X \cup \{z\}, & R' & = & R \cup \{(x, z)\} \cup \{(v, z) : vRx\}, \\ S' & = & S \cup \{(x, z)\}, & T' & = & T \cup \{(z, B)\} \end{array}$$

Otherwise, let y immediately succeed x in (X, R). If $B = T(y)$ set:

$$\begin{array}{lllllll} X' & = & X, & R' & = & R, \\ S' & = & S \cup \{(x, y)\} & T' & = & T. \end{array}$$

Otherwise, we have $B \mathrel{-\!\!3} T(y)$, and it suffices to fix $z \in W - X$ and set:

$$\begin{array}{lllllll} X' & = & X, & R' & = & R \cup \{(x, z), (z, y)\} \cup \\ & & & & & \cup \{(v, z) : vRx\} \cup \{(z, v) : yRv\}, \\ S' & = & S \cup \{(x, z)\}, & T' & = & T \cup \{z, B)\} \end{array}$$

Similarly, to kill a requirement of form (6) we use the mirror image of the Lemma above, proved using (A6b).

It is also necessary to check that when xSy we never need to insert a point between x and y in order to kill a requirement of form 8(1) or (2). Reviewing the construction of Section 3.2 above, this follows from parts (3), (4) of the Lemma above. The remaining details are left to the reader.

A total order is *discrete* if every element but the maximum (if any) has an immediate successor, and every element but the minimum (if any) has an immediate predecessor. The foregoing argument establishes that we get a complete axiomatisation for the tense logic of discrete total orders by adding to $\mathbf{L}_2$ the following weakened versions of (A6a, b):

$$p \wedge Hp \to G\bot \vee FHp, \quad p \wedge Gp \to H\bot \vee PGp.$$

A total order is *homogeneous* if for any two of its points x, y there exists an automorphism carrying x to y. Such an order cannot have a maximum or minimum and must be either dense or discrete. In Burgess [1979] it is indicated that a complete axiomatisation of the tense logic is homogeneous orders is obtainable by adding to $\mathbf{L}_4$ the following which should be compared with (A5a) and (A6a, b):

$$(Fp \to FFp) \vee [(q \wedge Hq \to FHq) \wedge (q \wedge Gq \to PGq)].$$

3.6 Continuity

A *cut* in a total order (X, R) is a partition (Y, Z) of X into two nonempty pieces, such that whenever $y \in Y$ and $z \in Z$ we have yRz. A *gap* is a cut (Y, Z) such that Y has no maximum and Z no minimum. (X, R) is complete if it has no gaps. The *completion* (X^+, R^+) of a total order (X, R) is the complete total order obtained by inserting, for each gap (Y, Z) in (X, R), an element $w(Y, Z)$ after all elements of Y and before all elements of Z. For example, the completion of the rational numbers in their usual order is the real numbers in their usual order. The extension $\mathbf{L}_7$ of $\mathbf{L}_2$ obtained by adding (A7a, b) is complete for the class $\mathcal{K}_7$ of complete total orders. The proof requires a couple of Lemmas:

LEMMA *Let T be a perfect chronicle on a total order (X, R), and (Y, Z) a gap in (X, R). Then if $G\alpha \in T(z)$ for all $z \in Z$, then $G\alpha \in T(y)$ for some $y \in Y$.*

Proof. Suppose for contradiction that $G\alpha \in T(z)$ for all $z \in Z$ but $F\neg\alpha \approx \neg G\alpha \in T(y)$ for all $y \in Y$. For any $y_0 \in Y$ we have $F\neg\alpha \wedge FG\alpha \in T(y)$. Hence, by A7a, $F(G\alpha \wedge HF\neg\alpha) \in T)y_0)$, and there is an x with y_0Rx and $G\alpha \in HF\neg\alpha \in T(x)$. But this is impossible, since if $x \in Y$ then $G\alpha \notin T(x)$, while if $x \in Z$ then $HF\neg\alpha \notin T(x)$. ■

LEMMA *Let T be a perfect chronicle on a total order (X, R). Then T can be extended to a perfect chronicle T^+ on its completion (X^+, R^+).*

Proof. For each gap (Y, Z) in (X, R), the set:

$$C(Y, Z) = \{P\alpha : \exists y \in Y(\alpha \in T(y))\} \cup \{F\alpha : \exists z \in Z(\alpha \in T(z))\}$$

is consistent. This is because any finite subset, involving only $y_1, \ldots, y_m$ form Y and $z_1, \ldots, z_n$ from Z will be contained in $T(x)$ where x is any element of Y after all the y_i or any element of Z before all the z_j. Hence, we can define a coherent chronicle T^+ on (X^+, R^+) by taking $T^+(w(Y, Z))$ to be some MCS extending $C(Y, Z)$. Now if $F\alpha \in T^+(w(Y, Z))$, we claim that $F\alpha \in T(z)$ for some $z \in Z$. For if not, then $G\neg\alpha \in T(z)$ for al $z \in Z$, and by the previous Lemma, $G\neg\alpha \in T(y)$ for some $y \in Y$. But then $PG\neg\alpha$, which implies $\neg F\alpha$, would belong to $C(Y, Z) \subseteq T^+(w(Y, Z))$, a contradiction. It hardly needs saying that if $F\alpha \in T(z)$, then there is some x with zRx and *a fortiori* $w(Y, Z)R^+x$ having $\alpha \in T(x)$. This shows T^+ is prophetic. Axiom (A7b) gives us a mirror image to the previous Lemma, which can be used to show T^+ historic. ■

To prove the completeness of $\mathbf{L}_7$ for $\mathcal{K}_7$, given a consistent γ_0 use the work of Section 2.2 above to construct a perfect chronicle T on a frame (X, R) such that $\gamma_0 \in T(x_0)$ for some x_0. Then use the foregoing Lemma to extend to a perfect chronicle on a complete total order, as required to prove satisfiability. ■

Similarly, $\mathbf{L_R}$, the extension of $\mathbf{L}_2$ obtained by adding (A4a, b) and (A5a) and (A7a, b) is complete for the class of complete dense total orders without maximum or minimum, sometimes called *continuous* orders. As a matter of fact, our construction shows that any formula consistent with this theory is satisfiable in the completion of the rationals, that is, in the reals. Thus $\mathbf{L_R}$ is the tense logic of real time and, hence, of the time of classical physics.

3.7 Well-Orders

The extension $\mathbf{L}_8$ of $\mathbf{L}_2$ obtained by adding (A8) is complete for the class $\mathcal{K}_8$ of all well-orders. For the proof it is convenient to introduce the abbreviations Ip for $Pp \vee p \vee Fp$ or 'p sometime', and Bp for $p \wedge \neg Pp$ or 'p for the first time'. an easy consequence of (A8) is $Ip \rightarrow IBp$: if something *ever* happens, then there is a *first* time when it happens the reader can check that the following are valid over total orders; hence, theses of ($\mathbf{L}_2$ and *a fortiori* of $\mathbf{L}_9$):

1. $Ip \wedge Iq \rightarrow I(Pp \wedge q) \vee I(p \wedge q) \vee I(p \wedge Pq)$,
2. $I(q \wedge Fr) \wedge I(PBp \wedge Bq) \rightarrow I(p \wedge Fr)$.

Now, understanding consistency, MCS, and related notions relative to $\mathbf{L}_8$, let δ_0 be any consistent formula and D_0 any MCS containing it. Let $\delta_1, \ldots, \delta_k$ be all the proper subformulas of δ_0. Let Γ be the set of formulas of form

$$(\neg)\delta_0 \wedge (\neg)\delta_1 \wedge \ldots \wedge (\neg)\delta_k$$

where each δ_i appears once, plain or negated. Note that distinct elements of Γ are truth-functionally inconsistent. Let $\Gamma' = \{\gamma \in \Gamma : I\gamma \in D_0\}$. Note that for each $\gamma \in \Gamma'$ we have $IB\gamma \in D_0$, and that for distinct $\gamma, \gamma' \in \Gamma'$ we must by (1) have either $I(PB\gamma \wedge B\gamma')$ or $I(PB\gamma' \wedge B\gamma)$ in D_0. Enumerate the elements of Γ' as $\gamma_0, \gamma_1, \ldots, \gamma_n$ so that $I(PB\gamma_i \wedge B\gamma_j) \in D_0$ iff $i < j$. We write $i \lhd j$ if $I(\gamma_i \wedge F\gamma_j) \in D_0$. This clearly holds whenever $i < j$, but may also hold in other cases. A crucial observation is:

$$(+) \qquad \text{If } i < j \leq k \text{ and } k \lhd i, \text{ then } j \lhd i$$

This follows from (2). These tedious preliminaries out of the way, we will now define a set X of ordinals and a function t from X to Γ'. Let $a, b, c, \ldots$ range over *positive* integers:

We put $0 \in X$ and set $t(0) = \gamma_0$.
If $0 \lhd 0$ we also put each $a \in X$ and set $t(a) = \gamma_0$.
We put $\omega \in X$ and set $t(\omega) = \gamma_1$.
If $1 \lhd 1$ we also put each $\xi = \omega \cdot b \in X$ and set $t(\xi) = \gamma_1$.
If $1 \lhd 0$ we also put each $\xi = \omega \cdot b + a \in X$ and set $t(\xi) = \gamma_0$.
We put $\omega^2 \in X$ and set $t(\omega^2) = \gamma_2$.
If $2 \lhd 2$ we also put each $\xi = \omega^2 \cdot c \in X$ and set $t(\xi) = \gamma_2$.
If $2 \lhd 1$ we also put each $\xi = \omega^2 \cdot c + \omega \cdot b \in X$, and set $t(\xi) = \gamma_1$.
If $2 \lhd$ we also put each $\xi = \omega^2 \cdot c + \omega \cdot b + a \in X$ and set $t(\xi) = \gamma_0$.
and so on.

Using (+) one sees that whenever $\xi, \eta \in X$ and $\xi < \eta$, then $i \lhd j$ where $t(\xi) = \gamma_i$ and $t(\eta) = \gamma_j$. Conversely, inspection of the construction shows that:

1. whenever $\xi \in X$ and $t(\xi) = \gamma_j$ and $j \lhd k$, then there is an $\eta \in X$ with $\xi < \eta$ and $t(\eta) = \gamma_k$
2. whenever $\xi \in X$ and $t(\xi) = \gamma_j$ and $i < j$, then there is an $\eta \in X$ with $\eta < \xi$ and $t(\eta) = \gamma_i$.

For $\xi \in X$ let $T(\xi)$ be the set of conjuncts of $t(\xi)$. Using (1) and (2) one sees that T satisfies all the requirements 8(1,2,3,4) for a perfect chronicle, *so far as these pertain to subformulas of* δ_0. Inspection of the proof of Lemma 9 then shows that this suffices to prove δ_0 satisfiable in the well-order $(X, <)$. ∎

Without entering into details here, we remark that variants of $\mathbf{L}_8$ provide axiomatisations of the tense logics of the integers, the natural numbers, and of finite total orders. In particular, for the natural numbers one uses $\mathbf{L}_\omega$, the extension of $\mathbf{L}_2$ obtained by adding (A8) and $p \wedge Gp \to H\bot \vee PGp$. $\mathbf{L}_\omega$ is the tense logic of the notion of time appropriate for discussing the working of a digital computer, or of the mental mathematical constructions of Brouwer's 'creative subject'.

3.8 Lattices

The extension $\mathbf{L}_9$ of $\mathbf{L}_1$ obtained by adding (A4a, b) and (A9a, b) is complete for the class $\mathcal{K}_9$ of partial orders without maximal or minimal elements in which any two elements have an upper and a lower bound. We sketch the modifications in the work of Section 3.2 above needed to prove this:

To begin with, we must revise clause 10(2) in the definition of M to read:

2g. R is a partial order on X having a maximum and a minimum.

This necessitates revisions in the proof of the Killing Lemma, for which the following will be useful:

LEMMA *Let A, B, C be* MCS*s. If $A \dashv\!\exists\, B$ and $A \dashv\!\exists\, C$, then there exists an* MCS *D such that $B \dashv\!\exists\, D$ and $C \dashv\!\exists\, D$.*

Proof. The problem quickly reduces to showing $\{\beta : G\beta \in B\} \cup \{\gamma : G\gamma \in C\}$ consistent. For this it suffices (using 3(4)) to show that $\beta \wedge \gamma$ is consistent whenever $G\beta \in B, G\gamma \in C$. Now in that case we have $FG\beta, FG\gamma \in A$, since $A \dashv\!\exists\, B, C$. By A9a, we then have $GF\beta \in A$, and by 3(2) we then have $F(F\beta \wedge G\gamma) \in A$ and $FF(\beta \wedge \gamma) \in A$, which suffices to prove $\beta \wedge \gamma$ consistent as required. ∎

Turning now to the Killing Lemma, trouble arises when for a given $(X, R, T) \in M$ a requirement of form Definition 8(1) is said to be 'killed' for some x other than the maximum y of (X, R) and some $F\gamma \in T(x)$.

Fixing an MCS B with $T(x) \dashv\!\exists\, B$ and $\gamma \in B$, and $az \in W - X$, we would like to ad z to x placing it after x and assigning it the MCS B. But we cannot simply do this, else the resulting partial order would have no maximum. (For y and z would be incomparable.) So we apply the Lemma (with $A = T(x), C = T(y)$) to obtain an MCS D with $B \dashv\!\exists\, D$ and $T(y) \dashv\!\exists\, D$. We fix a $w \in W - X$ distinct from z, and set:

$$\begin{aligned}
&X' = X \cup \{z, w\},\\
&R' = R \cup \{(x, z), (z, w)\} \cup \{(v, z) : vRx\} \cup \{(v, w) : v \in X\}.\\
&T' = T \cup \{(z, B), (w, D)\}.
\end{aligned}$$

Similarly, a requirement of form 8(2) involving an element other than the minimum is treated using the mirror image of the Lemma above, proved using (A9b).

Now given a formula γ_0 consistent with $\mathbf{L}_9$, the construction of Definition 10 above produces a perfect chronicle T on a partial order (X, R) with $\gamma_0 \in T(x_0)$ for some x_0. The work of Section 2.4 above shows that (X, R) will have no maximal or minimal elements. Moreover, (X, R) will be a union of partial orders (X_n, R_n) satisfying (2g). Then any $x, y \in X$ will have an R-upper bound and an R-lower bound, namely the R_n- maximumand R_n-minimum elements of any X_n containing them both. Thus, $(X, R) \in \mathcal{K}_9$ and γ_0 is satisfiable over $\mathcal{K}_9$. ■

A *lattice* is a partial order in which any two elements have a *least* upper bound and a *greatest* lower bound. Actually, our proof shows that $\mathbf{L}_9$ is complete for the class of lattices without maximum or minimum. It is worth mentioning that (A9a, b) could have been replaced by:

$$Fp \wedge Fq \to F(Pp \wedge Pq), \quad Pp \wedge Pq \to P(Fp \wedge Fq).$$

Weakened versions of these axioms can be used to give an axiomatisation for the tense logic of arbitrary lattices.

4 THE DECIDABILITY OF TENSE LOGICS

All the systems of tense logic we have considered so far are recursively decidable. Rather than give an exhaustive (and exhausting) survey, we treat here two examples, illustrating the two basic methods of proving decidability: one method, borrowed from modal logic, is that of using so- called *filtrations* to establish what is known as the *finite model property.* The other, borrowed from model theory, is that of using so-called *interpretations* in order to be able to exploit a powerful theorem of [Rabin, 1966].

THEOREM 13. $\mathbf{L}_9$ *is decidable.*

Proof. Let $\mathcal{K}$ be the class of models of (B1) and (B9a, b); thus $\mathcal{K}$ is like $\mathcal{K}_9$ except that we do *not* require antisymmetry. Let $\mathcal{K}'$ be the class of finite elements of $\mathcal{K}$. It is readily verified that $\mathbf{L}_9$ is sound for $\mathcal{K}$ and *a fortiori* for $\mathcal{K}'$. We claim that $\mathbf{L}_9$ is complete for $\mathcal{K}'$. This provides an effective procedure for testing whether a given formula α is a thesis of $\mathbf{L}_9$ or not, as follows: search simultaneously through all deductions in the system $\mathbf{L}_9$ and through all members of $\mathcal{K}'$—or more precisely, of some nice countable subclass of $\mathcal{K}'$ containing at least one representative of each isomorphism-type. Eventually one either finds a deduction of α, in which case α is a thesis, or one finds an element of $\mathcal{K}'$ in which $\neg\alpha$ is satisfiable, in which case by our completeness claim, α is not a thesis.

To prove our completeness claim, let γ_0 be consistent with $\mathbf{L}_9$. We showed in Section 2.9 above how to construct a perfect chronicle T on a frame $(X, R) \in \mathcal{K}_9 \subseteq \mathcal{K}$ having $\gamma_0 \in T(x_0)$ for some x_0. For $x \in X$, let $t(x)$ be the set of subformulas of γ_0 in $T(x)$. Define an equivalence relation on X by:

$$x \leftrightarrow y \text{ iff } t(x) = t(y).$$

Let $[x]$ denote the equivalence class of x, X' the set of all $[x]$. Note that X' is finite, having no more than 2^k elements, where k is the number of subformulas of γ_0. Consider the relations on X' defined by:

$$\begin{array}{lll} aR^+b & \text{iff} & xRy \text{ for some } x \in a \text{ and } y \in b, \\ aR'b & \text{iff} & \text{for some finite sequence } a = c_0, c_1, \ldots, c_{n-1}, c_n = b \\ & & \text{we have } c_iR^+c_{i+1} \text{ for all } i < n. \end{array}$$

Clearly R' is transitive, while R^+ and, hence, R' inherit from R the properties expressed by B9a, b. Thus $(X', R') \in \mathcal{K}'$. Define a function t' on X' by letting $t'(a)$ be the common value of $t(x)$ for all $x \in a$. In particular for $a_0 = [x_0]$ we have $\gamma_0 \in t'(a_0)$. We claim that t' satisfies clauses 8(1, 2, 3, 4) of the definition of a perfect chronicle *so far as these pertain to subformulas of* γ_0. As remarked in Section 3.8 above, this suffices to show γ_0 satisfiable in (X', R') and, hence, satisfiable over $\mathcal{K}'$ as required.

In connection with Definition 8(1), what we must show is:

1. whenever $F\gamma \in t(a)$ there is a b with $aR'b$ and $\gamma \in t(b)$

Well, let $a = [x]$, so $F\gamma \in t(x) \subseteq T(x)$. There is a y with xRy and $\gamma \in t(y)$ since T is prophetic. Letting $b = [y]$ we have aR^+b and so $aR'b$.

In connection with Definition 8(3) what we must show is:

3′. whenever $G\gamma \in t(a)$ and $aR'b$, then $\gamma \in t(b)$.

For this it clearly suffices to show:

3^+ whenever $G\gamma \in t(a)$ and aR^+b, then $\gamma \in t(b)$ and $G\gamma \in t(b)$.

To show this, assuming the two hypotheses, fix $x \in a$ and $y \in b$ with xRy. We have $G\gamma \in t(x) \subseteq T(x)$, so by (A1a), $GG\gamma \in T(x)$. Hence, $\gamma \in t(y)$ and $G\gamma \in t(y)$, since T is coherent—which completes the proof.

Definitions 8(2, 4) are treated similarly. ■

THEOREM 14. $\mathbf{L}_R$ *is decidable.*

Proof. We introduce an alternative definition of *validity* which is useful in other contexts. To each tense-logical formula α we associate a first-order formula $\hat{\alpha}$ as follows: for a sentential variable p_i we set $\hat{p}_i = P_i(x)$ where P_i is a one-place predicate variable. We then proceed inductively:

$$(\neg\alpha)\hat{} = \neg\hat{\alpha},$$
$$(\alpha \wedge \beta)\hat{} = \hat{\alpha} \wedge \hat{\beta}$$
$$(G\alpha)\hat{} = \forall y(x < y \to \hat{\alpha}(y/x)),$$
$$(H\alpha)\hat{} = \forall y(y < x \to \hat{\alpha}(y/x)).$$

Here (y/x) represents the result of substituting for x the alphabetically first variable y not occurring yet. Given a valuation V in a frame (X, R) we have an interpretation in the sense of first-order model theory, in which R interprets the symbol $<$ and $V(p_i)$ the symbol P_i. Unpacking the definitions it is entirely trivial that we always have:

$$(*) \qquad a \in V(\alpha) \text{ iff } (X, R, V(p_0), V(p_1), V(p_2), \ldots) \vDash \hat{\alpha}(x),$$

where $\vDash$ is the usual satisfaction relation of model theory. We now further define:

$$a^+ = \forall P_0 \forall P_1, \ldots, \forall P_k \forall x \hat{\alpha}(x),$$

where $p_0, p_1, \ldots, p_k$ include all the variables occurring in α. Note that α^+ is a second-order formula of the simplest kind: it is *monadic* (all its second-order variables are *one*- place predicate variables) and *universal* (consisting of a string of universally-quantified second-order variables prefixed to a first-order formula). It is entirely trivial that:

$$(+) \qquad \alpha \text{ is valid in } (X, R) \text{ iff } (X, R) \vDash \alpha^+$$

It follows that to prove the decidability of the tense logic of a given class $\mathcal{K}$ of frames it will suffice to prove the decidability of the set of universal monadic (second-order) formulas true in all members of $\mathcal{K}$.

Let $2^{<\omega}$ be the set of all finite $0, 1$-sequences. Let $*0$ be the function assigning the argument $s = (i_0, i_1, \ldots, i_n) \in 2^{<\omega}$ the value $s * 0 = (i_0, i_1, \ldots, i_n, 0)$, and similarly for $*1$. Rabin proves the decidability of the set $S2S$ of monadic (second order) formulas true in the structure $(2^{<\omega}, *0, *1)$. He deduces as an easy corollary the decidability of the set of monadic formulas true in the frame $(\mathbb{Q}, <)$ consisting of the rational numbers with their usual order. This immediately yields the decidability of the system $\mathbf{L}_Q$ of Section 2.5 above. Further corollaries relevant to tense logic are the decidability of the set of monadic formulas true in all countable total orders, and similarly for countable well-orders.

It only remains to reduce the decision problem for $\mathbf{L}_R$ to that for $\mathbf{L}_Q$. The work of 2.7 above shows that a formula α is satisfiable in the frame $(\mathbb{R}, <)$ consisting of the real numbers with their usual order, iff it is satisfiable in the frame $(\mathbb{Q}, <)$ by a valuation V with the property:

1. $V(\alpha) = \mathbb{Q}$ for every substitution instance α of (A7a or b).

Inspection of the proof actually shows that it suffices to have:

2. $V(\alpha') = \mathbb{Q}$ where α' is the conjunction of al instances of (A7a or b) obtainable by substituting subformulas of α for variables.

A little thought shows that this amounts to demanding:

3. $V(\alpha \wedge GH\alpha') \neq \varnothing$.

In other words, α is satisfiable in $(\mathbb{R}, <)$ iff $\alpha \wedge GH\alpha'$ is satisfiable in $(\mathbb{R}, <)$, which effects the desired reduction. For the lengthy original proof see [Bull, 1968]. Other applications of Rabin's theorem are in [Gabbay, 1975]. Rabin's proof uses automata-theoretic methods of Büchi; these are avoided by [Shelah, 1975]. ■

5 TEMPORAL CONJUNCTIONS AND ADVERBS

5.1 Since, Until, Uninterruptedly, Recently, Soon

All the systems discussed so far have been based on the primitives $\neg, \wedge, G, H$. It is well-known that any truth function can be defined in terms of $\neg, \wedge$. Can we say something comparable about temporal operators and G, H? When this question is formulated precisely, the answer is a resounding NO.

DEFINITION 15. Let φ be a first-order formula having one free variable x and no nonlogical symbols but the two-place predicate $<$ and the one-place predicates $P_1, \ldots, P_n$. corresponding to φ we introduce a new n-place connective, the (first-order, one-dimensional) temporal operator $O(\varphi)$. We describe the formal semantics of $O(\varphi)$ in terms of the alternative approach of Theorem 14 above: we add to the definition of ˆ the clause:

$$(O(\varphi)(\alpha_1, \ldots, \alpha_n))\hat{} = \varphi(\hat{\alpha}_1/P_1, \ldots, \hat{\alpha}_n/P_n).$$

Here $\hat{\alpha}/P$ denotes substitution of the formula $\hat{\alpha}$ for the predicate variable P. We then let formula (*) of Theorem 14 above *define* $V(\alpha)$ for formulas α involving $O(\varphi)$. Examples 16 below illustrate this rather involved definition. If $\mathcal{O} = \{O(\varphi_1), \ldots, O(\varphi_k)\}$ is a set of temporal operators, an *$\mathcal{O}$-formula* is one built up from sentential variables using $\neg, \wedge$, and elements of $\mathcal{O}$. A temporal operator $O(\varphi)$ is *$\mathcal{O}$- definable* over a class $\mathcal{K}$ of frames if there is an $\mathcal{O}$- formula α such that $O(\varphi)(p_1, \ldots, p_n) \leftrightarrow \alpha$ is valid over $\mathcal{K}$. $\mathcal{O}$ is *temporally complete* over $\mathcal{K}$ if every temporal operator is $\mathcal{O}$-definable over $\mathcal{K}$. Note that the smaller $\mathcal{K}$ is—it may consist of a single frame— the easier it is to be temporally complete over it.

EXAMPLES 16.

1. $\forall y(x < y \to P_1(y))$
2. $\forall y(y < x \to P_1(y))$
3. $\exists y(x < y \wedge \forall z(x < z \wedge z < y \to P_1(z)))$
4. $\exists y(y < x \wedge \forall z(y < z \wedge z < x \to P_1(z)))$
5. $\exists y(x < y \wedge P_1(y) \wedge \forall z(y < z \wedge z < x \to P_1(z)))$

For (1), $O(\varphi)$ is just G. For (2), $O(\varphi)$ is just H. For (3), $O(\varphi)$ will be written G', and may be read 'p is going to be uninterruptedly the case for some time'. For (4), $O(\varphi)$ will be written H', and may be read 'p has been uninterruptedly the case for some time. For (5), $O(\varphi)$ will be written U, and $U(p, q$ may be read 'until p, q'; it predicts a future occasion of p's being the case, up until which q is going to be uninterruptedly the case. For (6), $O(\varphi)$ will be written S, and $S(p, q)$ may be read 'since p, q'. In terms of G' we define $F' = \neg G; \neg$, read 'p is going to be the case arbitrarily soon'. In terms of H' we define $P' = \neg H' \neg$, read 'p has been the case arbitrarily recently'. Over all frames, Gp is definable as $\neg U(\neg p, \top)$, and G' as $U(\top, p)$. Similarly, H and H' are definable in terms of S. The following examples are due to H. Kamp:

PROPOSITION 17. *G' is not G, H-definable over the frame* $(\mathbb{R}, <)$.

Sketch of Proof. Define two valuations over that frame by:

$$V(p) = \{0, \pm 1, \pm 2, \pm 3, \ldots\} \quad W(p) = V(p) \cup \{\pm\tfrac{1}{2}, \pm\tfrac{1}{4}, \pm\tfrac{1}{8}, \ldots\}$$

Then intuitively it is plausible, and formally it can be proved that for any G, H-formula α we have $0 \in V(\alpha)$ iff $0 \in W(\alpha)$. But $0 \in V(G'p) - W(G'p)$. ■

PROPOSITION 18. *U is not G, H, G', H'-definable over the frame* $(\mathbb{R}, <)$.

Sketch of Proof. Define two valuations by:

$$\begin{aligned} &V(p) = \{\pm 1, \pm 2, \pm 3, \pm 4, \ldots\} \quad W(p) = \{\pm 2, \pm 3, \pm 4, \ldots\} \\ &V(q) = W(q) = \text{ the union of the open intervals} \\ &\qquad \ldots, (-5, -4), (-3, -2), (-1, +1), \\ &\qquad (+2, +3), (+4, +5), \ldots \end{aligned}$$

Then intuitively it is plausible, and formally it can be proved that for any G, H, G', H'-formula α we have $0 \in V(\alpha)$ iff $0 \in W(\alpha)$. But $0 \in V(U(p, q) - W(U(p, q))$. ■

Such examples might inspire pessimism, but [Kamp, 1968] proves:

THEOREM 19. *The set $\{U, S\}$ is temporally complete over continuous orders.*

We will do no more than outline the difficult proof (in an improved version due to Gabbay): Let $\mathcal{O}$ be a set of temporal operators, $\mathcal{K}$ a class of frames. An $\mathcal{O}$-formula α is *purely past* over $\mathcal{K}$ if whenever $(X, R) \in \mathcal{K}$ and $x \in K$ and V, W are valuations in (X, R) agreeing before x (so that for all i, $V(p_i) \cap \{y : yRx\} = W(p_i) \cap \{y : yRx\}$) then $x \in V(\alpha)$ iff $x \in W(\alpha)$. Similarly, one defines *purely present* and *purely future*, and one defines *pure* to mean purely past, or present, or future. Note that $Hp, H'p, S(p, q)$, are purely past, their mirror images purely future, and any truth-functional compound of variables purely present. $\mathcal{O}$ has the *separation property* over $\mathcal{K}$ if for every $\mathcal{O}$-formula α there exists a truth- functional compound β of $\mathcal{O}$-formulas pure over $\mathcal{K}$ such that $\alpha \leftrightarrow \beta$ is valid over $\mathcal{K}$. $\mathcal{O}$ is *strong* over $\mathcal{K}$ if G, H are $\mathcal{O}$-definable over $\mathcal{K}$. Gabbay [1981a] proves:

Criterion 20. *Over any given class $\mathcal{K}$ of total orders, if $\mathcal{O}$ is strong and has the separation property, then it is temporally complete.*

A full proof being beyond the scope of this survey (see, however, the next chapter 'Advanced Tense Logic'), we offer a sketch: we wish to find for any first-order formula $\varphi(x, <, P_1, \ldots, P_n)$ an $\mathcal{O}$-formula $\alpha(p_1, \ldots, p_n)$ *representing* it in the sense that for any $(X, R) \in \mathcal{K}$ and any valuation V and any $a \in X$ we have:

$$a \in V(\alpha) \text{ iff } (X, R, V(p_1), \ldots, V(p_n) \vDash \varphi(a/x).$$

The proof proceeds by induction on the depth of nesting of quantifiers in φ, the key step being $\varphi(x) = \exists y \psi(x, y)$. In this case, the atomic subformulas of ψ are of the forms $P_i(x), P_i(z), z < x, z = x, x < z, z = w, z < w$, where z and w are variables other than x. Actually, we may assume there are no subformulas of the form $P_i(x)$ since these can be brought outside the quantifier $\exists y$. We introduce new singulary predicates Q^-, Q^0, Q^+ and replace the subformulas of ψ of forms $z < x, z = x, x < z$ by $Q^-(z), Q^0(z), Q^+(z)$, to obtain a formula $\vartheta(y, <, P_1, \ldots, P_n, Q^-, Q^0, Q^+)$ to which we can apply our induction hypothesis, obtaining an $\mathcal{O}$-formula $\delta(p_1, \ldots, p_n, q^-, q^0, q^+)$ representing it. Let $\gamma(p_1, \ldots, sp_n) = \delta(p-1, \ldots, p_n, Fq, q, Pq)$, and $\beta = P\gamma \vee \gamma \vee F\gamma$. It is readily verified that for any $(X, R) \in \mathcal{K}$ and any $a, b \in X$ and any valuation V with $V(q) = \{a\}$ that we have:

$$\begin{array}{lll} b \in V(\gamma) & \text{iff} & (X, R, V(p_1), \ldots, V(p_n)) \vDash \psi(a/x, b/y), \\ a \in V(\beta) & \text{iff} & (X, R, V(p_1), \ldots, V(p_n)) \vDash \varphi(a/x). \end{array}$$

By hypothesis, β is equivalent over $\mathcal{K}$ to a truth-functional compound of purely past formulas β_i^-, purely present ones β_j^0, and purely future ones

β_k^+. In each β_i^- (resp. β_j^0) (resp. β_k^+) replace q by $\perp$ (resp. $\top$) (resp. $\perp$) to obtain an $\mathcal{O}$-formula α. It is readily verified that α represents φ.

It 'only' remains to show:

LEMMA 21. *The set $\{U, S\}$ has the separation property over complete orders.*

Though a full proof is beyond the scope of this survey, we sketch the method for achieving the separation for a formula α in which there is a single occurrence of an S within the scope of a U. This case (and its mirror image) is the first and most important in a general inductive proof.

To begin with, using conjunctive and disjunctive normal forms and such easy equivalences as:

$$U(p \vee q, t) \leftrightarrow U(p, t) \vee U(q, t),$$
$$U(p, q \wedge r) \leftrightarrow U(p, q) \wedge U(p, r),$$
$$\neg S(q, r) \leftrightarrow S(\neg r, \neg q) \vee P'\neg r,$$

we can achieve a reduction to the case where α has one of the forms:

1. $U(p \wedge S(q, r), t)$
2. $U(p, q \wedge S(r, t))$

For (1), an equivalent which is a truth-functional compound of pure formulas is provided by :

$1'$. $[(S(q, r) \vee q) \wedge U(p, r \wedge t)] \vee U(q \wedge U(p, r \wedge t), t)$

For (2) we have:

$2'$. $\{[(S(r, t) \wedge t) \vee r] \wedge [U(p, t) \vee U(\beta, t)]\} \vee \beta$

where β is: $F'\neg t \wedge U(p, q \vee S(r, t))$. This, despite its complexity, is purely future. The observant reader should be able to see how completeness is needed for the equivalence of (2) and w').

Unfortunately, U and S take us no further, for Kamp proves:

PROPOSITION 22. *The set $\{U, S\}$ is not temporally complete over* $(\mathbb{Q}, <)$.

Without entering into details, we note that one undefinable operator is $O(\varphi)$ where φ says:

$$\exists y(x < y \wedge \forall z(x < z \wedge z < y \rightarrow$$
$$(\forall w(x < w \wedge w < z \rightarrow P_1((w)) \vee \forall w(z < w \wedge w < y \rightarrow P_2(w)))))$$

Over complete orders $O(\varphi)(p, q)$ amounts to $U(G'q \wedge (p \vee q), p)$.

J. Stavi has found two new operators U', S' and proved:

THEOREM 23. *The set $\{U, S, U', S'\}$ is temporally complete over total orders.*

Gabbay has greatly simplified the proof: the idea is to try to prove the separation property over arbitrary total orders, and see what operators one needs. One quickly hits on the right U', S'. The combinatorial details cannot detain us here.

What about axiomatisability for U, S-tense logic? Some years ago Kamp announced (but never published) finite axiomatisability for various classes of total orders. Some are treated in [Burgess, 1982], where the system for dense orders takes a particularly simple form: we depart from standard format only to the extent of taking U, S as our primitives. As characteristic axioms, it suffices to take the following and their mirror images:

$$
\begin{aligned}
&G(p \to q) \to (U(p,r) \to U(q,r)) \wedge ((U(r,p) \to U(r,q)) \\
&p \wedge U(q,r) \to U(q \wedge S(p,r), r), \\
&U(p,q) \leftrightarrow U(p, q \wedge U(p,q)) \leftrightarrow U(q \wedge U(p,q), q), \\
&U(,q) \wedge \neg U(p,r) \to U(q \wedge \neg r, q), \\
&U(p,q) \wedge U(r,s) \to U(p \wedge r, q \wedge s \vee U(p \wedge s, q \wedge s) \vee U(q \wedge r, q \wedge s).
\end{aligned}
$$

A particularly important axiomatisabiity result is in [Gabbay *et al.*, 1980].

What about decidability? Rabin's theorem applies in most cases, the notable exceptions being complete orders, continuous orders, and $(\mathbb{R}, <)$. Here techniques of monadic second-order logic are useful. Decidability for the cases of complete and continuous orders is established in [Gurevich, 1977, Appendix]; and for $(\mathbb{R}, <)$ in [Burgess and Gurevich, 1985]. A fact (due to Gurevich) from the latter paper worth emphasising is that the U, S-tense logics of $(\mathbb{R}, <)$ and of arbitrary continuous orders are *not* the same.

5.2 Now, Then

We have seen that simple G, H-tense logic is inadequate to express certain temporal operators expressible in English. Indeed it turns out to be inadequate to express even the shortest item in the English temporal vocabulary, the word 'now'. Just what role this word plays is unclear— some incautious writers have even claimed it is semantically redundant— but [Kamp, 1971] gives a thorough analysis. Let us consider some examples:

0. The seismologist predicted that there would be an earthquake.
1. The seismologist predicted that there would be an earthquake *now*.
2. The seismologist predicted that there would already have been an earthquake *before now*.
3. The seismologist predicted that there would be an earthquake, but not till *after now*.

As Kamp says:

> The function of the word ‘now’ in (1) is to make the clause to which it applies—i.e. ‘there would be an earthquake’—refer to the moment of utterance of (1) and not to the moment of moments (indicated by other temporal modifiers that occur in the sentence) to which the clause would refer (as it does in (0)) if the word ‘now’ were absent.

5.3 Formal Semantics

To formalise this observation, we introduce a new one-place connective J (for *jetzt*). We define a *pointed frame* to be a frame with a designated element. A *valuation* in a pointed frame (X, R, x_0) is just a valuation in $(X < R)$. We extend the definition of 0.4 above to G, H, J-formulas by adding the clause:

$$V(J\alpha) = X \text{ if } x_0 \in V(\alpha), \varnothing \text{ if } x_0 \notin V(\alpha)$$

is *valid* in (X, R, x_0) if $x_0 \in V(\alpha)$ for all valuations V.

An alternative approach is to define a 2-*valuation* in a frame (X, R) to b a function assigning each p_i a subset of the Cartesian product X^2. Parallel to 1.4 above we have the following inductive definition:

$$V(\neg\alpha) = X^2 - V(\alpha),$$
$$V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta),$$
$$V(G\alpha) = \{(x, y) : \forall x'(xRx' \to (x', y) \in V(\alpha)\},$$
$$V(H\alpha), \text{ similarly},$$
$$V(J\alpha) = \{(x, y) : (y, y) \in V(\alpha)\}$$

α is *valid* in (X, R) if $\{(y, y) : y \in X\} \subseteq V(\alpha)$ for all 2-valuations V.

The two alternatives are related as follows: Given a 2-valuation V in the frame (X, R), for each $y \in X$ consider the valuation V_y in the pointed frame $(X < R, y)$ given by $V_y(p_i) = \{x : (x, y) \in V(p_i)\}$. Then we always have $(y, y) \in V(\alpha)$ iff $y \in V_y(\alpha)$.

The second approach has the virtue of making it clear that though J is not a temporal operator in the sense of the preceding section, it is in a sense that can be made precise a *two-dimensional* tense operator. This suggests the project of investigating two-and multi-dimensional operators generally. Some such operators, for instance the ‘then’ of [Vlach, 1973], have a natural reading in English. Among other items in our bibliography, [Gabbay, 1976] and [Gabbay and Guenthner, 1982] contain much information on this topic.

Using J we can express (0)–(3) as follows:

$0'$. P (seismologist says: F (earthquake occurs)),

$1'$. P (seismologist says: J (earthquake occurs)), 2′. P (seismologist says: JP (earthquake occurs)),

3′. P (seismologist says: JF (earthquake occurs)).

The observant reader will have noted that (0′)–(3′) are not really representable by G, H, J-formulas since they involve the notion of 'saying' or 'predicting'), a *propositional attitude*. Gabbay, too, gives many examples of uses of 'now' and related operators, and on inspection these, too, turn out to involve propositional attitudes. That this is no accident is shown by the following result of Kamp:

THEOREM 24 (Eliminability theorem). *For any G, H, J-formula α there is a G, H-formula α^* equivalent over all pointed frames.*

Proof. Call a formula *reduced* if it contains no occurrence of a J within the scope of a G or an H. Our first step is to find for each formula α an equivalent reduced formula α_R. This is done by induction on the complexity of α, only the cases $\alpha = G\beta$ or $\alpha = H\beta$ being nontrivial. In, for instance, the latter case, we use the fact that any truth-function can be put into disjunctive normal form, plus the following valid equivalence:

$$(R) \qquad {}_H((Jp \wedge q) \wedge r) \leftrightarrow ((Jp \wedge H(q \vee r)) \vee (\neg Jp \wedge Hr))$$

Details are left to the reader. Our second step is to observe that if β is reduced, then it is equivalent to the result β^- of dropping all its occurrences of J. It thus suffices to set $\alpha^* = (\alpha_R)^-$. ■

The foregoing theorem says that in the presence of truth-functions and G and H, the operator J is, in a sense, redundant. By contrast, examples (0)–(3) suggest that in contexts with propositional attitudes, J is *not* redundant; the lack of a generally-accepted formalisation of the logic of propositional attitudes makes it impossible to turn this suggestion into a rigorous theorem. But in contexts with *quantifiers*, Kamp *does* prove rigorously that J is irredundant. Consider:

4. The Academy of Arts rejected an applicant who was to become a terrible dictator and start a great war.

5. The Academy of arts has rejected an applicant who is to become a terrible dictator and start a great war.

The following formalisations suggest themselves:

4′. $P(\exists x(R(x) \wedge FD(x))$

5′. $P(\exists x(R(x) \wedge JFD(x))$,

the difference between (4) and (5) lying precisely in the fact that the latter, unlike the former, definitely places the dictatorship and war in the hearer's future. What Kamp proves is that ($5'$) cannot be expressed by a G, H-formula with quantifiers.

Returning to sentential tense logic, Theorem 24 obviously reduces the *decision* problem for G, H, J-tense logic to that for G, H- tense logic. As for *axiomatisability*, obviously we cannot adopt the standard format of G, H-tense logic, since the rule TG does not preserve validity for G, H, J-formulas. For instance:

(D0) $p \leftrightarrow Jp$

is valid, but $G(p \leftrightarrow Jp)$ and $H(p \leftrightarrow Jp)$ are not. Kamp overcomes this difficulty, and shows how, in very general contexts, to obtain from a complete axiomatisation of a logic without J, a complete axiomatisation of the same logic with J. For the sentential G, H, J- tense logic of total orders, the axiomatisation takes a particularly simple form: take as sole rule MP. Let Lp abbreviate $Hp \wedge p \wedge Gp$. Take as axioms all substitution instances of tautologies, of (Do) above, and of $L\alpha$, where α may be any item on the lists (D1), (D2) below, or the mirror image of such an item:

(D1) $G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq)$
$p \rightarrow GPp$
$Gp \leftrightarrow GGp$
$Lp \leftrightarrow GHp$

(D2) $J\neg p \leftrightarrow \neg Jp$
$J(p \wedge q) \leftrightarrow Jp \wedge Jq$
$\neg L\neg Jp \leftrightarrow LJp$
$Lp \rightarrow Jp.$

(In outline, the proof of completeness runs thus: using (D1) one deduces $Lp \rightarrow LLp$. It follows that the class of theses deducible without use of (D0) is closed under TG. Our work in Section 3.2 shows that we then get the complete G, H-tense logic of total orders. We then use (D2) to prove the equivalence (R) in the proof of Theorem 24 above. More generally, for any $\alpha, \alpha \leftrightarrow \alpha_R$ is deducible without using (D0). Moreover, using D0, $\beta \leftrightarrow \beta^-$ is deducible for any reduced formula β. Thus in general $\alpha \leftrightarrow \alpha^*$ is a thesis, completing the proof.)

6 TIME PERIODS

The geometry of Space can be axiomatised taking unextended *points* as basic entitites, but it can equally well be axiomatised by taking as basic certain regular open solid regions such as *spheres*. Likewise, the order of

Time can be described either (as in Section 1.1) in terms of *instants* in terms of *periods* of non zero duration. Recently it has become fashionable to try to redo tense logic, taking periods rather than instants as basic. Humberstone [1979] seems to be the first to have come out in print with such a proposal. This approach has become so poplar that we must give at least a brief account of it; further discussion can be found in [van Benthem, 1991]. (See also Kuhn's discussion in the last chapter of this Volume of the *Handbook*.)

In part, the switch from instants to periods is motivated by a desire to model certain features of natural language. One of these is *aspect*, the verbal feature which indicates whether we are thinking of an occurrence as an *event* whose temporal stages (if any) do not concern us, or as a protracted *process*, forming, perhaps the backdrop for other occurrences. These two ways of looking at death (a popular, if morbid, example) are illustrated by:

> When Queen Anne died, the Whigs brought in George.
>
> While Queen Anne was dying, the Jacobites hatched treasonable plots.

Another feature of linguistic interest is the peculiar nature of *accomplishment* verbs, illustrated by:

1. The Amalgamated Conglomerate Building was built during the period March–August 1972.

1′. The ACB was built during the period April–July, 1972.

2. The ACB was being built (i.e. was under construction) during the period March–August, 1972.

2′. The ACB was under construction during the period April– July, 1972.

Note that (1) and (1′) are inconsistent, whereas (2) implies (2′)!

In part, the switch is motivated by a philosophical belief that periods are somehow more basic than instants. This motivation would be more convincing were 'periods' not assumed (as they are in too many recent works) to have sharply-defined (i.e. *instantaneous*) beginnings and ends. It may also be remarked that at the level of *experience* some occurrences do *appear* to be instantaneous (i.e. we don't *discern* stages in them). Thus 'bubbles when they burst' *seem* to do so 'all at once and nothing first'. While at the level of *reality*, some occurrences of the sort studied in quantum physics may well take place instantaneously, just as some elementary particles may well be pointlike. Thus the philosophical belief that every occurrence takes some time (period) to occur is not *obviously* true on any level.

Now for the mechanics of the switch: for any frame (X, R) we consider the set $I(X, R)$ of nonempty bounded open intervals of form $\{z : xRz \wedge zRy\}$.

Among the many relations on this set that could be defined in terms of R we single out two:

$$\begin{array}{llll} \textit{Inclusion}: & a \subseteq b & \text{iff} & \forall x(x \in a \to x \in b), \\ \textit{Order}: & a \lhd b & \text{iff} & \forall x \forall y(x \in a \wedge y \in b \to xRy). \end{array}$$

To any class $\mathcal{K}$ of frames we associate the class $\mathcal{K}'$ of those structures of form $(I(X,R), \subseteq, \lhd)$ with $(X < R) \in \mathcal{K}$, and the class $\mathcal{K}^+$ of those structures (Y, S, T) that are isomorphic to elements of $\mathcal{K}'$.

A first problem in switching from instants to periods as the basis for the logic of time is to find each important class $\mathcal{K}$ of frames a set of postulates whose models will be precisely the structures in $\mathcal{K}^+$. For the case of dense total orders without extrema, and for some other cases, suitable postulate sets are known, though none is very elegant. Of course this first problem is not yet a problem of *tense* logic; it belongs rather to applied first- and second-order logic.

To develop a period-based *tense* logic we define a *valuation* in a structure (Y, S, T)—where S, T are binary relations on Y—to be a function V assigning each p_i a subset of Y. Then from among all possible connectives that could be defined in terms of S and T, we single out the following:

$$\begin{array}{l} V(\neg\alpha) = Y - V(\alpha) \\ V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta) \\ V(\nabla\alpha) = \{a : \forall b(bSa \to b \in V(\alpha))\} \\ V(\Delta\alpha) = \{a : \forall b(aSb \to b \in V(\alpha))\} \\ V(F\alpha) = \{a : \exists b(aTb \wedge b \in V(\alpha))\} \\ V(P\alpha) = \{a : \exists b(bTa \wedge b \in V(\alpha))\}. \end{array}$$

The main *technical* problem now is, given a class $\mathbf{L}$ of structures (Y, S, T)—for instance, one of form $\mathbf{L} = \mathcal{K}^+$ for some class $\mathcal{K}$ of frames—to find a sound and complete axiomatisation for the tense logic of $\mathbf{L}$ based on the above connectives. Some results along these lines have been obtained, but none as definitive as those of instant-based tense logic reported in Section 3. Indeed, the choice of relations ($\subseteq$ and $\lhd$), and of admissible classes $\mathbf{L}$ (should we only consider classes of form $\mathcal{K}^+$?), and of connectives $(\neg, \wedge, \Delta, \nabla, F, P)$, and of admissible valuations (should we impose restrictions, such as requiring $b \in V(p_i)$ whenever $a \in V(p_i)$ and $b \subseteq a$?) are all matters of controversy.

The main problem of *interpretation*—one to which advocates of period-based tense logic have perhaps not devoted sufficient attention—is how to make intuitive sense of the notion $a \in V(p)$ of a sentence p being true *with respect to* a time-period a. One proposal is to take this as meaning that p is true *throughout* a. Now given a valuation W in a frame (X, R), we can define a valuation $I(W)$ in $I(X, R)$ by $I(W)(p_i) = \{a : a \subseteq W(p_i)\}$. When and *only* when V has the form $I(W)$ is 'p is true throughout a' a tenable reading of $a \in V(p)$. It is not, however, easy to characterise intrinsically those V that admit a representation in the form $V = I(W)$. Note that even in this case, $a \in V(\neg p)$ does not express ‘$(\neg p)$ is true throughout a’ (but rather ‘$\neg(p$ is true throughout $a)$’). Nor does $a \in V(p \vee q)$ express ‘$(p \vee q)$ is true throughout a’.

Another proposal, originating in [Burgess, 1982] is to read $a \in V(p)$ as ‘$^{+}_{+}$ is *almost* always true during a’. This reading is tenable when V has the form $J(W)$ for some valuation W in (X, R), where $J(W)(p_i)$ is by definition $\{a : a - W(p_i)$ is nowhere dense in the order topology on $(XR)\}$. In this case, ‘$(\neg p)$ is almost always true during a’ is expressible by $a \in V(\nabla\neg p)$, and ‘$(p \vee q)$ is almost always true during a; by $a \in V(\nabla\neg\nabla\neg(p \vee q))$. But the whole problem of interpretation for period-based tense logic deserves more careful thought.

There have been several proposals to redo tense logic on the basis of 3- or 4- of multi-valued truth-functional logic. It is tempting, of instance, to introduce a truth-value ‘unstatable’ to apply to, say, ‘Bertrand Russell is smiling’ in 1789. In connection with the switch from instants to periods, some have proposed introducing new truth-values ‘changing from true to false’ and ‘changing from false to true’ to apply to, say, ‘the rocket is at rest’ at take-off and landing times. Such proposals, along with proposals to combine, say, tense logic and intuitionistic logic, lie beyond the scope of this survey.

7 GLIMPSES AROUND

7.1 Metric Tense Logic

In *metric* tense logic we assume Time has the structure of an ordered Abelian group. We introduce variables $x, y, z, \ldots$ ranging over group elements, and simples $0, +, <$ for the group identity, addition, and order. We introduce operators $\mathcal{F}, \mathcal{P}$ joining terms for group elements with formulas. Here, for instance, $\mathcal{F}(x + y)(p \wedge q)$ means that it will be the case $(x + y)$ time-units hence that p and q. Metric tense logic is intended to reflect such ordinary-language *quantitative* expressions as ‘10 years from now’ or ‘tomorrow about this time’ or ‘in less than five minutes’. The *qualitative* F, P of nonmetric tense logic can be recovered by the definitions $Fp \leftrightarrow \exists x > 0\mathcal{F}xp, Pp \leftrightarrow \exists x > 0\mathcal{P}xp$. Actually, the ‘ago’ operator $\mathcal{P}$ is definable in terms of the ‘hence’ operator $\mathcal{F}$ since $\mathcal{P}xp$ is equivalent to $\mathcal{F} - xp$. It is not hard to write down axioms for metric tense logic whose completeness can be proved by a Henkin-style argument.

But decidability is lost: the decision problem for metric tense logic is easily seen to be equivalent to that for the set of all universal monadic (second- order) formulas true in all ordered Abelian groups. We will show that the decision problem for the validity of first-order formulas involving a single two-place predicate $\in$—which is well known to be unsolvable—is reducible to the latter: given a first-order $\in$-formula φ, fix two one-place predicate variables U, V. Let φ_0 be the result of restricting all quantifiers in φ to U (i.e. $\forall x$ is replaced $\forall x(U(x) \to \ldots)$ and $\exists x$ by $\exists x(U(x) \wedge \ldots)$.) Let φ_1 be the result of replacing each atomic subformula $x \in y$ of φ_0 by $\exists z(V(z) \wedge V(z+x) \wedge V(z+x+y))$. Let φ_2 be the universal monadic formula $\forall U \forall V(\exists x U(x) \to \varphi_2)$. Clearly if φ is logically valid, then so is φ_2 and, in particular, the latter is true in all ordered Abelian groups. If φ is not logically valid, it has a countermodel consisting of the positive integers equipped with a binary relation E. Consider the product $\mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z}$ is the additive group of integers; addition in this group is defined by $(x,y)+(x',y') = (x+x', y+y')$; the group is orderable by $(x,y) < (x',y')$ iff $x < x'$ or ($x = x'$ and $y < y'$). Interpret U in this group as $\{(n,0) : n > 0\}$; interpret V as the set consisting of the $(2^m 3^n, 0), (2^m 3^n, m)$ and $(2^m 3^n, m+n)$ for those pairs (m,n) with mEn. This gives a countermodel to the truth of φ_2 in $\mathbb{Z} \times \mathbb{Z}$. Thus the desired reduction of decision problems has been effected.

Metric tense logic is, in a sense, a hybrid between the 'regimentation' and 'autonomous tense logic' approaches to the logic of time. Other hybrids of a different sort—not easy to describe briefly—are treated in an interesting paper of [Bull, 1978].

7.2 Time and Modality

As mentioned in the introduction, Prior attempted to apply tense logic to the exegesis of the writings of ancient and mediaeval philosophers and logicians (and for that matter of modern ones such as C. S. Peirce and J. Łukasiewicz) on future contingents. The relations between tense and mode or modality is properly the topic of Richmond H. Thomason's chapter in this volume.

We can, however, briefly consider here the topic of so-called *Diodorean* and *Aristotelian* modal fragments of a tense logic L. The former is the set of modal formulas that become theses of L when $\Box p$ is defined as $p \wedge Gp$; the latter is the set of modal formulas that becomes theses of L when $\Box p$ is defined as $Hp \wedge p \wedge Gp$. Though these seem far-fetched definitions of 'necessity', the attempt to isolate the modal fragments of various tense logics undeniable was an important stimulus for the earlier development of our subject. Briefly the results obtained can be tabulated as follows. It will be seen that the modal fragments are usually well-known C. I. Lewis systems.

Class of frames	Tense logic	Diodorean fragment	Aristotelian fragment
All frames	$\mathbf{L}_0$	$\mathbf{T}(=\mathbf{M})$	**B**
Partial orders	$\mathbf{L}_1$	**S4**	**B**
Lattices	$\mathbf{L}_0$	**S4.2**	**B**
Total orders	$\mathbf{L}_2, \mathbf{L}_5$	**S4.3**	**S5**
Dense orders			

The Diodorean fragment of the tense logic $\mathbf{L}_6$ of discrete orders has been determined by M. Dummett; the Aristotelian fragment of the tense logic of trees has been determined by G. Kessler. See also our comments below on R. Goldblatt's work.

7.3 Relativistic Tense Logic

The *cosmic* frame is the set of all point-events of space-time equipped with the relation of *causal accessibility*, which holds between u and v if a signal (material or electromagnetic) could be sent from u to v. The $(n+1)$-dimensional *Minkowski* frame is the set of $(n+1)$-tuples f real numbers equipped with the relation which holds between $(a_{0,1}, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ iff:

$$\sum_{i-1}^{n}(b_n - a_n)^2 - (b_0 - a_0)^2 > 0 \text{ and } b_0 > a_0.$$

For present purposes, the content of the *special theory of relativity* is that the cosmic frame is isomorphic to the 4-dimensional Minkowski frame.

A little calculating shows that any Minkowski frame is a lattice without maximum or minimum, hence the tense logic of special relativity should at least include $\mathbf{L}_0$. Actually we will also want some axioms to express the density and continuity of a Minkowski frame. A surprising discovery of Goldblatt [1980] is that the *dimension* of a Minkowski frame influences its tense logic. Indeed, he sows that for each n there is a formula γ_{n+1} which is valid in the $(m+1)$-dimensional Minkowski frame iff $m < n$. For example, writing Ep for $p \wedge Fp, \gamma_2$ is:

$$\begin{aligned}&Ep \wedge Eq \wedge Er \wedge \neg E(p \wedge q) \wedge \neg E(p \wedge r) \wedge \neg E(q \wedge r) \rightarrow \\ &E((Ep \wedge Eq) \vee (Ep \wedge Er) \vee (Eq \wedge Er)).\end{aligned}$$

On the other hand, he also shows that the dimension of a Minkowski frame does *not* influence the diodorean modal fragment of its tense logic: the Diodorean modal logic of special relativity is the same as that of arbitrary lattices, namely **S4.2**. Combining Goldblatt's argument with the 'trousers world' construction in general relativity, should produce a proof that the Diodorean modal fragment of the latter is the same as that of arbitrary partial orders, namely **S4**.

Despite recent advances, the tense logic of special relativity has not yet been completely worked out; that of general relativity is even less well understood. Burgess [1979] contains a few additional philosophical remarks.

7.4 Thermodynamic Time

One of the oldest metaphysical concepts (found in Hindu theology and pre-Socratic philosophy, and in modern psychological dress in Nietzsche and celestial mechanical dress in Poincaré) is that everything that has ever happened is destined to be repeated over and over again. This leads to a degenerate tense logic containing the principles $Gp \to Hp$ and $Gp \to p$ among others.

An antithetical view is that traditionally associated with the Second Law of Thermodynamics, according to which irreversible change are taking place that will eventually drive the Universe to a state of 'heat-death', after which no further change on a macroscopically observable level will take place. The tense logic of this view, which raises several interesting technical points, has been investigated by S. K. Thomason [1972]. The first thing to note is that the principle:

(A10) $GFp \to FGp$

is acceptable for p expressing propositions about macroscopically observable states of affairs provided these do not contain hidden time references; e.g. p could be 'there is now no life on Earth', but not 'particle κ currently has a momentum of precisely k gram- meters/second' or 'it is now an even number of days since the Heat Death occurred'. For the antecedent of (A20) says that arbitrarily far in the future there will be times when p is the case. But for the p that concern us, the truth-value of p is never supposed to change after the Heat Death. So in that case, there will come a time after which p is always going to be true, in accordance with the consequent of (A10).

The question now arises, how can we *formalise* the restriction of p to a special class of sentences? In general, propositions are represented in the formal semantics of tense logic by subsets of X in a frame (X, R). A restricted class of propositions could thus be represented by a distinguished family $\mathcal{B}$ of subsets of X. This motivates the following definition: an *augmented* frame is a triple $(X, R, \mathcal{B})$ where (X, R) is a frame, $\mathcal{B}$ a subset of the lower set $\mathcal{B}(X)$ of X closed under complementation, finite intersection, and the operations:

$$\begin{aligned} gA &= \{x \in X : \forall y \in X(xRy \to y \in A)\} \\ hA &= \{x \in X : \forall y \in X(yRx \to y \in A)\}. \end{aligned}$$

A *valuation* in $(X, R, \mathcal{B})$ is a function V assigning each variable p_i an element of $\mathcal{B}$. The closure conditions on $\mathcal{B}$ guarantee that we will then have $V(\alpha) \in \mathcal{B}$

for *all* formulas α. It is now clear how to define validity. Note that if $\mathcal{B} = \mathcal{P}(X)$, then the validity in $(X, R, \mathcal{B})$ reduces to validity in (X, R); otherwise more formulas may be valid in the former than the latter.

It turns out that the extension $\mathbf{L}_{10}$ of $\mathbf{L_Q}$ obtained by adding (A10) is (sound and) complete for the class of augmented frames $(X, R, \mathcal{B})$ in which (X, R) is a dense total order without maximum or minimum and:

$$\forall B \in \mathcal{B} \exists x (\forall y (xRy \to y \in B) \vee \forall y (xRy \to y \notin B)).$$

We have given complete axiomatisations for many intuitively important classes of frames. We have not yet broached the questions: when does the tense logic of a given class of frames admit a complete axiomatisation? Wen does a given axiomatic system of tense logic correspond to some class of frames in the sense of being complete for that class? For information on these large questions, and for bibliographical references, we refer the reader to Johan van Benthem's chapter in Volume 3 of this edition of the *Handbook* on so-called 'Correspondence Theory'. Suffice it to say here that positive general theorems are few, counterexamples many. The thermodynamic tense logic $\mathbf{L}_{10}$ exemplifies one sort of pathology. Though it is not inconsistent, there is *no* (unaugmented) frame in which all its theses are valid!

7.5 Quantified Tense Logic

The interaction of temporal operators with universal and existential quantifiers raises many difficult issues, both philosophical (over identity through changes, continuity, motion and change, reference to what no longer exists or does not exist, essence, and many, many more) and technical (over undecidability, nonaxiomatisability, undefinability or multi-dimensioal operators, and so forth) that it is pointless to attempt even a survey of the subject in a paragraph or tow. We therefore refer the reader to Nino Cocchiarella's chapter in this volume and James W. Garson's chapter in Volume 3 of this edition of the *Handbook*, both on this subject.

Princeton University, USA.

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