Subject 24.244. Modal Logic. Answers to the second p-set.

1. KT5 is the smallest normal modal system that contains (T) and (5).

a) Show, by giving a derivaion, that every instance of schema (B) is in KT5.

1.	$(\Diamond\varphi \to \Box\Diamond\varphi)$	(5)
2.	$(\varphi \to \Diamond\varphi)$	Dual of (T)
3.	$(\varphi \to \Box\Diamond\varphi)$	TC 1, 2

b) Show, by giving a derivation, that every instance of schema (4) is in KT5.

1.	$(\Box\varphi \to \Box\Diamond\Box\varphi)$	(B), from 1a)
2.	$(\Diamond\Box\varphi \to \Box\varphi)$	Dual of (5)
3.	$\Box(\Diamond\Box\varphi \to \Box\varphi)$	Nec 2
4.	$(\Box\Diamond\Box\varphi \to \Box\Box\varphi)$	K 3
5.	$(\Box\varphi \to \Box\Box\varphi)$	TC 1, 4

c) Show that any binary relation that is reflexive and Euclidean is transitive and symmetric.
Suppose that R is reflexive and Euclidean. Suppose that Rwv. Because R is reflexive Rww. Consequently, because R is Euclidean Rvw, and R is symmetric.

Now suppose Rwv and Rvu. By symmetry, Rvw. By the Euclidean property, Ruw.

In the version of this p-set I orginally published on Canvas, I left off the requirement that R be reflexive. Without this requirement, R needn't be symmetric. Here is Peter Hart's example: W = {*a*,b,c}. R = {<*a,*b>, <*a*,c>, <b,c>, <c,b>}.

2. KB4 is the smallest normal modal system that contains (B) and (4)

a) Show, by giving a derivation, that every instance of schema (5) is in KB4.

1.	$(\Diamond\varphi \to \Box\Diamond\Diamond\varphi)$	(B)
2.	$(\Diamond\Diamond\varphi \to \Diamond\varphi)$	Dual of (4)
3.	$\Box(\Diamond\Diamond\varphi \to \Diamond\varphi)$	Nec 2
4.	$(\Box\Diamond\Diamond\varphi \to \Box\Diamond\varphi)$	TC 1,3

b) Show that any binary relation that is symmetric and transitive is Euclidean.
Suppose that R is symmetric and transitive and that Rwv and Rwu. By symmetry, we have Rvw. By transitivity, Rvw.

3. K45 is the smallest normal modal system that contains (4) and (5).

a) Show that a modal system can be transitive and Euclidean without being symmetric.
The example from 1c): W = {*a*,b,c}. R = {<*a,*b>, <*a*,c>, <b,c>, <c,b>}.

b) Show that "$(P \to \Box\Diamond P)$" isn't in K45. In the frame from 3a), let "P" be true in *a* only. The "$\Diamond P$" isn't true in b, so "$\Box\Diamond P$" isn't true in *a*.

4. Show that every modal formula can be derived in the system obtained from Strict S5 by incorporating Substitution as an additional rule, in addition to (MP) and (Nec).

1.	$\Diamond\alpha_1$	Axiom of Strict S5
2.	$\Diamond\Box\varphi$	Subst 1
3.	$(\Diamond\Box\varphi \to \varphi)$	Dual of (B)
4.	φ	TC 2, 3

5. **Find English sentences that you can substitute for "P" and "Q" so that, when "◊" is translated as "it is logically possible that," the English translation of "(P ∧ Q) ∧ ~ ◊(P ∧ ~ Q) ∧ ~ ◊(~ P ∧ Q) ∧ ◊(~ P ∧ ~ Q)" comes out true. Explain your answer.**
   The algorithm says to substitute "P" for "P" and "((Q ∨ P) ∧ ~ ~ P)" for "Q," but we can simplify by substituting "P" for "P." Translate it into English by substituting "The Heat beat the Celtics" for both "P" and "Q." (The Miami Heat beat the Boston Celtics in six games.)

6. **Find English sentences that you can substitute for "P" and "Q" so that, when "◊" is translated as "it is logically possible that," the English translation of "◊(P ∧ Q) ∧ ◊(P ∧ ~ Q) ∧ ~ ◊(~ P ∧ Q) ∧ (~ P ∧ ~ Q)" comes out true. Explain your answer.**

   The algorithm says to substitute "P" for "P" and "((Q ∨ ⊥) ∧ ~ ~ P)" for "Q," but the latter can be simplified to "(Q ∧ P)." Translate "P" as "The Celtics beat the Heat" and "Q" as "The Raptors beat the Celtics." (The Celtics beat the Toronto Raptors in seven games.)

7. **Let's say that two sentences φ and ψ are Strict-S5-equivalent iff (φ ↔ ψ) is in Strict S5. How many nonempty Strict-S5-equivalence classes of modal formulas containing no atomic formulas other than "P," "Q," and "R" are there? You don't need to list them. Just say how many there are. Explain your answer.** In Strict S5, every state description is possible, so all we need do to specify a canonical model is to say which of the eight state descriptions is true. That is, there are eight model descriptions. Each sentence is Strict-S5 equivalent to a disjunction of model descriptions, so that, up to Strict-S5 equivalence, there are $2^8 = 256$ sentences.

8. **Let's say that two sentences φ and ψ are S5-equivalent iff (φ ↔ ψ) is in S5. How many nonempty S5-equivalence classes of modal formulas containing no atomic formulas other than "P," "Q," and "R" are there? Explain your answer.**
   To specify a model, there are 2^3 choices of which state description is actual, and 2^7 choices of which of the remaining state descriptions are possible. There are thus $2^{10} = 1024$ model descriptions. Every sentence is equivalent to disjunction of model descriptions, so that, up to S5-equivalence, there are 2^{1024} sentences.