Self-reference Lemma. For any formula $\varphi(x)$, there is a sentence ψ so that PA $\vdash (\psi \leftrightarrow \varphi([\ulcorner \psi \urcorner])$.

Proof: Let d be the following calculable function:

If m has the form $\theta(x)$, $d(m) = \ulcorner \theta([m])$.

Otherwise $d(m) = 0$.

d is calculable, so there is a formula $\mathfrak{D}$ that functionally represents it. Let $\chi(x)$ be the formula $(\exists y)(\mathfrak{D}(x,y) \wedge \psi(y))$, let $m = \ulcorner\chi(x)\urcorner$, and let $\varphi = \chi([m])$, so that $\ulcorner \varphi \urcorner = d(m)$. In PA, we can prove the following:

$(\forall y)(\mathfrak{D}([m],y) \leftrightarrow y = [\ulcorner \varphi \urcorner])$.

$((\exists y)(\mathfrak{D}([m],y) \wedge \psi(y)) \leftrightarrow \psi([\ulcorner \varphi \urcorner]))$.

$(\varphi \leftrightarrow \psi([\ulcorner \varphi \urcorner]))$.

Write $y\ B_\Gamma\ x$ for "y is the code of a sequence of sentences containing x, each of which is either a member of Γ, an axiom of logic, or obtained from earlier members of the sequence by modus ponens."

$\mathrm{Bew}_\Gamma(x) =_{\mathrm{Def}} (\exists y)\ y\ B_\Gamma\ x$, i.e., x is provable in Γ

Use the self-reference lemma to find γ so that PA $\vdash (\gamma \leftrightarrow \sim \mathrm{Bew}_\Gamma([\gamma]))$.
γ asserts its own unprovability.

If Γ is consistent, γ is unprovable.

If Γ is ω-consistent, γ is unrefutable.

The proof of "$\mathrm{CON}(\Gamma) \rightarrow \sim \mathrm{Bew}_\Gamma([\gamma])$" can be formalized in PA giving us the

Second Incompleteness Theorem. If Γ is consistent, $\Gamma \nvdash \mathrm{CON}(\Gamma)$