Self-reference Lemma. For any formula $\varphi(x)$, there is a sentence ψ so that PA $\vdash (\psi \leftrightarrow \varphi([\ulcorner \psi \urcorner]))$.

Proof: Let d be the following calculable function:

If m has the form $\theta(x)$, $d(m) = \ulcorner \theta([m])$.

Otherwise $d(m) = 0$.

d is calculable, so there is a formula $\mathfrak{D}$ that functionally represents it. That is, for each n,

$$\text{PA} \vdash (\forall y)(\mathfrak{D}(n],y) \leftrightarrow y = [d(n)]).$$

Let $\chi(x)$ be the formula $(\exists y)(\mathfrak{D}(x,y) \wedge \psi(y))$.
Let m = $\ulcorner\chi(x)\urcorner$/
Let $\varphi = \chi([m])$, so that $\ulcorner \varphi \urcorner = d(m)$. In PA, we can prove:

$$(\forall y)(\mathfrak{D}([m],y) \leftrightarrow y = [\ulcorner \varphi \urcorner]).$$

$$((\exists y)(\mathfrak{D}([m],y) \wedge \psi(y)) \leftrightarrow \psi([\ulcorner \varphi \urcorner])).$$

$$(\varphi \leftrightarrow \psi([\ulcorner \varphi \urcorner])).$$

First Incompleteness Theorem.
Write y B_Γ x for "y is the code of a sequence of sentences containing x, each of which is either a member of Γ, an axiom of logic, or obtained from earlier members of the sequence by modus ponens."
B_Γ is decidable in PA.

$\text{Bew}_\Gamma(x) =_{\text{Def}} (\exists y)\ y\ B_\Gamma\ x$, i.e., x is provable in Γ

Use the self-reference lemma to find γ so that PA $\vdash (\gamma \leftrightarrow \sim \text{Bew}_\Gamma([\gamma]))$.
γ asserts its own unprovability.

If Γ is consistent, γ is unprovable.

If Γ is ω-consistent, γ is unrefutable.

This shows that, if Γ is ω-consistent, it's incomplete.
Rosser strengthened this to: If Γ is consistent, it's incomplete.