Second Incompleteness Theorem.

We showed: If Γ is consistent, $\Gamma \nvdash \gamma$.
Expressed in terms of the codes: $\mathrm{CON}(\Gamma) \rightarrow \sim \mathrm{Bew}_{\Gamma}([\ulcorner \gamma \urcorner])$.
Formalizing this proof within Γ:

$$\Gamma \vdash (\mathrm{CON}(\Gamma) \rightarrow \sim \mathrm{Bew}_{\Gamma}([\ulcorner \gamma \urcorner]))$$
$$\Gamma \vdash (\mathrm{CON}(\Gamma) \rightarrow \gamma)$$

If Γ proves $\mathrm{CON}(\Gamma)$, Γ proves γ.
But we know that, if Γ proves γ, Γ is inconsistent.
Therefore, if Γ is consistent, it doesn't prove $\mathrm{CON}(\Gamma)$.

Löb's Theorem. For Γ a recursively axiomatized extension of PA, $(\mathrm{Bew}([\ulcorner \varphi \urcorner]) \rightarrow \varphi)$ is provable in Γ if and only if φ is provable in Γ.
Proof: (⇐) is obvious
This proof of **(⇒)** is due to Kripke.

Suppose φ isn't provable in Γ.
So $\Gamma \cup \{\sim \varphi\}$ is consistent.
By the second incompleteness theorem, $\mathrm{CON}(\Gamma \cup \{\sim \varphi\})$ isn't provable in $\Gamma \cup \{\sim \varphi\}$.

Reformulate this arithmetically:

$$\Gamma \cup \{\sim \varphi\} \nvdash \sim \mathrm{Bew}_{\Gamma \cup \{\sim\varphi\}}([\ulcorner \sim 0 = 0 \urcorner]).$$
$$\Gamma \cup \{\sim \varphi\} \nvdash \sim \mathrm{Bew}_{\Gamma}([\ulcorner (\sim \varphi \rightarrow \sim 0 = 0) \urcorner]).$$
$$\Gamma \cup \{\sim \varphi\} \nvdash \sim \mathrm{Bew}_{\Gamma}([\ulcorner \varphi \urcorner]).$$
$$\Gamma \nvdash (\sim \varphi \rightarrow \sim \mathrm{Bew}_{\Gamma}([\ulcorner \varphi \urcorner])).$$
$$\Gamma \nvdash (\mathrm{Bew}_{\Gamma}([\ulcorner \varphi \urcorner]) \rightarrow \varphi).$$

We can formalize this argument:

$$\Gamma \vdash \mathrm{Bew}_{\Gamma}([\ulcorner (\mathrm{Bew}_{\Gamma}([\ulcorner \varphi \urcorner]) \rightarrow \varphi) \urcorner]).$$