Löb conditions:

(L1) If $\Gamma \vdash \varphi$, PA $\vdash \mathrm{Bew}_\Gamma([\ulcorner \varphi \urcorner])$.

(L2) $\Gamma \vdash (\mathrm{Bew}_\Gamma(\ulcorner \varphi \urcorner) \rightarrow \mathrm{Bew}_\Gamma([\ulcorner \mathrm{Bew}_\Gamma([\ulcorner \varphi \urcorner]) \urcorner]))$.

(L3) $\Gamma \vdash (\mathrm{Bew}_\Gamma([\ulcorner (\varphi \rightarrow \psi) \urcorner]) \rightarrow (\mathrm{Bew}_\Gamma([\ulcorner \varphi \urcorner]) \rightarrow \mathrm{Bew}_\Gamma([\ulcorner \psi \urcorner])))$.

Translating Gödel and Löb's Results into Modal Terms

Fix Γ a recursively axiomatized extension of PA, writing "Bew" instead of "Bew_Γ."

A *Γ-interpretation* is a function *i* that associates an arithmetical sentence with each modal formula, subject to the following constraints:

$$i(\varphi \vee \psi) = (i(\varphi) \vee i(\psi))$$
$$i(\varphi \wedge \psi) = (i(\varphi) \wedge i(\psi))$$
$$i(\varphi \rightarrow \psi) = (i(\varphi) \rightarrow i(\psi))$$
$$i(\sim \varphi) = \sim i(\varphi)$$
$$i(\Box \varphi) = \mathrm{Bew}_\Gamma([\ulcorner i(\varphi) \urcorner])$$

A modal formula φ is *always provable* in Γ iff, for each Γ-interpretation i, $i(\varphi)$ is provable in Γ. φ is *always true* for Γ iff, for each Γ-interpretation i, $i(\varphi)$ is true in the standard model $\mathbb{N}$.

GL (= K4L) is the smallest set of modal formulas containing these axioms and closed under these rules:

Axioms. (K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

(4) $\Box\varphi \rightarrow \Box \Box\varphi$.

(L) $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi)$

Rules: Necessitation.

Tautological Consequence.

Solovay's Theorem. Assuming Γ doesn't prove any Σ sentence that are false in the standard model, a model formula is in GL if and only if it's always provable.