Subject 24.244. Modal Logic. Answers to the third p-set.

1. **Which of the following formulas are in S4? If the formula is in S4, give a derivation. If it isn't in S4, give a transitivs4e, reflexive model in which it is false.**

a) **($\Box P \supset \Box\Diamond\Box P$).** In S4.

1. $(\Box\sim\Box P \supset \sim\Box P)$	(T)
2. $(\Box P \supset \Diamond\Box P)$	TC, 1
3. $\Box(\Box P \supset \Diamond\Box P)$	Nec 2
4. $(\Box\Box P \supset \Box\Diamond\Box P)$	From 3 by (K)
5. $(\Box P \supset \Box\Box P)$	(4)
6. $(\Box P \supset \Box\Diamond\Box P)$	TC 4, 5

b) **($\Diamond P \supset \Diamond\Box\Diamond P$).** Not in S4.
Take a model with two worlds, the actual world @ and w. @ has access to itself and w, whereas w has access only to itself. "P" is true in @ only. So "$\Diamond P$" is true in @ only, and "$\Box\Diamond P$" is false in both worlds.

c) **($\Box(P \vee \Box Q) \equiv (\Box P \vee \Box\Box Q)$).** Not in S4.
Take a model with two worlds, the actual world @ and w. @ has access to itself and w, whereas w has access only to itself. "P" is true only in @ and "Q" is true only in w. Then "$(P \vee \Box Q)$" is true in every world, so "$\Box(P \vee \Box Q)$" is true in @, although neither "$\Box P$" nor "$\Box\Box Q$" is true in @.

2. **True or false? Explain your answer: The following are equivalent, for any formula φ:**

a) **φ is true in every model $<W,R,I,@>$ with R a reflexive, transitive, symmetic relation on W.**

b) **φ is true in every model $<W,R,I,@>$ with R equal to $W \times W$.**

True. That a) implies b) is easy. $W \times W$ is reflexive, transitive, and symmetric. For the other direction, suppose that φ is true in every model in which every world has access to every world, and consider a model $<W,R,I,@>$ in which R is reflexive, transitive, and symmetric. Let W* consist of all the worlds in W that are accessible from @. Because R is reflexive, @ is an element of W*. Let I* be the function we get from I by restricting its domain to {atomic sentences} $\times$ W*. I claim that, for any formula ψ and any world w in W*, ψ is true in $<W^*,W^* \times W^*, I^*, w>$ if and only if it is true in $<W,R,I,w>$. This will give us the outcome we want, that φ is true in $<W,R,I,@>$.

We prove the claim by induction on the complexity of formulas. That it holds for atomic formulas is immediate from the way I* was defined. A conjunction is true in in $<W^*,W^* \times W^*, I^*,w>$ iff both conjuncts are true in $<W^*,W^* \times W^*,I^*,w>$ iff both conjuncts are true in $<W,R,I,@>$ (by inductive hypothesis) iff the conjunction is true in $<W,R,I,@>$. Similarly for the other Boolean connectives. We need to show that, for any $w \in W^*$, $\Box\psi$ is true in $<W,R,I,w>$ iff it's true in $<W^*,W^*\times W^*,I^*,w>$. W prove the two direction separately:

(⇒) Assume $\Box\psi$ is true in $<W,R,I,w>$. Take any world v in W*. Since w and v are both in W*, we hve R@w and R@v. By symmetry and transitivity, we have Rwv, and so ψ is true in $<W,R,I,v>$. By inductive hypothesis, ψ is true in $<W^*,W^*\times W^*,I^*,v>$. Since v was an arbitrary member of W*, we know that ψ is true in $<W^*,W^*\times W^*,I^*>$ in every world in W*, and hence

$\Box\psi$ is true in <W*,W*×W*,I*,w>.

(⇐) Assume that $\Box\psi$ is true in <W*,W*×W*,I*,w>. Take any world v in W with Rwv. Since w in in W*, we have R@w. Because R is transitive, R@v, and so v in in W*. So ψ is true in <W*,W*×W*,I*,v>. By inductive hypothesis, ψ is true in <W,R,I,v>. Since was an arbitrary member of W with Rwv, $\Box\psi$ must be true in <W,R,I,w>.

3. A binary relation R on a set W is a partial function iff, whenever Rwu and Rwv, we have u=v. Give a system of axioms that generates all the formulas true in every model <W,R,I,@> in which R is a partial function, and show that it works.

The set of formulas true in every world in which the accessibility relation is a partial function is an normal modal system, and it includes all instances of the schema

(P) $(\Diamond\varphi \supset \Box\varphi)$.

It therefore includes all formulas in KP, the smallest normal modal system that includes (P), whose axioms are (K) and (P) and whose rules are tautological consequence and necessitation.

We want to prove the converse, that any formula that's not in KP is false in some model whose accessibility relation is a partial function. Any sentence that's not in KP is false in some world in the canonical frame for KP, so it will be enough to show that the accessibility relaton on the canonical frame for KP is a partial function. For use in problem 4, we'll show something slightly stronger, namely, that the accessibility relation on the canonical frame for any normal modal system that includes KP is a partial function. Suppose that w, v, and u are in the canonical frame and Rwu and Rwv. If a formula φ is in v, $\Diamond\varphi$ is in w, so $\Box\varphi$ is in w, so φ is in u. If a formula ψ isn't in v, $\sim\psi$ is in v, so $\sim\psi$ is in u, so ψ isn't in u. So v = u.

4. A binary relation R on a set W is a total function iff it's a partial function whose domain is all of W. Give a system of axioms that generates all the formulas true in every model <W,R,I,@> in which R is a total function, and show that it works.

Instances of the axiom schema

(D) $(\Box\,\varphi \rightarrow \Diamond\varphi)$

are true at every world in every model in which the domain of the accessibility relation includes every world. Thus the set of formulas true in every world in every formula in which the accessibility relation is a total function includes both (P) and (D), so it includes the normal modal system KPD. To see that the set of formulas true in every world in which the accessibility relation in a total function is identical to KPD, it will suffice to show that the canonical frame for KPD is a total function. We know from problem 3 that it's a partial function, and any world that wasn't in the domain of the accessibility relation would be a world in which the instance "$(\Box\top \rightarrow \Diamond\top)$" of (D) was false.

5. Show that there is an algorithm for testing whether a formula is true in every model <W,R,I,@> in which R is a total function.

We know from problem 4 that, if a formula is true in every total-function model, it's derivable in KPD, and we can show it's in KPD by giving a derivation. Our basic plan is this: If a formula isn't true in every total-function model, we'll show this by displaying a total-function model in which it's false. The trouble is that, whereas we know from problem 4 that the canonical frame for KPD has contains a world in which χ is false, the canonical frame will contain infinitely many worlds and an attempt to verify that χ is false in @ might get bogged down forever, checking the status of various sentences in various worlds. Our revised plan is to start with a total-function model in which χ is false and trim it down to a total-function model with finitely many worlds in which χ is still false.

Define the *modal depth* of a formula as follows: An atomic formula has modal depth 0. A formula formed using the SC connectives will has as its modal depth the maximum of the modal depths of its components. The modal depth of $\Box\varphi$ will be 1 + the modal depth of φ. Let's say that the modal depth of χ is n. There is a model <W,R,I,@> in which χ is false. Consider the sequence of world $R^0(@)$, $R^1(@)$, $R^2(@)$, $R^3(@)$,..., where $R^0(w) = w$ and $R^{k+1}(w) = R(R^k(w))$. We'll see that what's going on in the worlds after $R^n(@)$ won't make any difference to the status of χ in @. We can clip off the sequence after $R^n(@)$ by altering the accessibility function so that it takes $R^n(@)$ back to $R^n(@)$ and still have a total-function model in which χ is false. Only now it's a model with finitely many worlds, so we can write it out as evidence that χ isn't valid.

Given a total-function model <W,R,I,@> in which χ is false, let's produce a finite total-function model <W*,R*,I*,@>, as follows:

> $W^* = \{R^0(@), R^1(@), R^2(@),..., R^n(@)\}$.
> $R^*(R^k(@) = R^{k+1}(@)$, for $k < n$.
> $R^*(R^n(@)) = R^n(@)$.
> I* is I restricted to {atomic sentences} × W*.
>
> **Claim.** If $w \in \{R^0(@), R^1(@),..., R^{n-k}(@)\}$, where $0 \leq k \leq n$, and the formula φ has modal degree $\leq k$, φ is true in <W*,R*,I*,w> iff φ is true in <W,R,I,w>.

We prove this by induction on k. The base case, k = 0, pretty much automatic, since I* and I agree where both are defined. Assume as inductive hypothesis that the claim holds for k, where $0 \leq k < n$. We want to show it holds for k+1. Take $w \in \{R^0(@), R^1(@),..., R^{n-(k+1)}\}$ and φ of modal degree $\leq$ k+1.The cases where φ is atomic, a disjunction, a conjunction, or a negation are effortless. We have to worry about the case where φ has the form $\Box\psi$. ψ has modal degree $\leq$ k. $\Box\psi$ is true in <W*,R*,I*,w> iff ψ is true in <W*,R*,I*,R(w)> iff ψ is true in <W,R,I,R(w)> [by inductive hypothesis] iff $\Box\psi$ is true in <W,R,I,w>.

From the claim, we see that χ is true in <W*,R*,I*,@> iff it's true in <W,R,I,@>.

6. Which of the systems KT, K4, KT4, and KT5 contain the formula "$((\Diamond P \wedge \Diamond Q) \leftrightarrow \Diamond(\Diamond P \wedge \Diamond Q))$"? Explain your answers.

It's not in KT. Take a model with two worlds, @, w, and v, with R = {<@,@>,<w,w>, <v,v>, <@,w>, <w,v>}, and with "P" and "Q" true in v only. "$(\Diamond P \wedge \Diamond Q)$" is true in w, so "$\Diamond(\Diamond P \wedge \Diamond Q)$" is true in @, but "$(\Diamond P \wedge \Diamond Q)$" isn't true in @.

It's not in K4. Take a model with two world, @ and w, with R = {<@,w>} and with "P" and "Q" true in w only. No formula that begins with "$\Diamond$" is true in w, so "$\Diamond(\Diamond P \wedge \Diamond Q)$" is false in @, although “$(\Diamond P \wedge \Diamond Q)$” is true in @.

It’s in KT4.

1.	$((\Diamond P \wedge \Diamond Q) \rightarrow \Diamond(\Diamond P \wedge \Diamond Q))$	Dual of (T)
2.	$(\sim \Diamond P \rightarrow \sim (\Diamond P \wedge \Diamond Q))$	(Taut)
3.	$\Box(\sim \Diamond P \rightarrow \sim(\Diamond P \wedge \Diamond Q))$	Nec 2
4.	$(\Box\sim\Diamond P \rightarrow \Box\sim(\Diamond P \wedge \Diamond Q))$	K 3
5.	$(\Diamond(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond \Diamond P)$	TC 4, Def. of “$\Diamond$”
6.	$(\Diamond \Diamond P \rightarrow \Diamond P)$	Dual of (4)
7.	$(\Diamond(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond P)$	TC 5, 6
8.	$(\sim \Diamond Q \rightarrow \sim(\Diamond P \wedge \Diamond Q))$	(Taut)
9.	$\Box(\sim \Diamond Q \rightarrow \sim(\Diamond P \wedge \Diamond Q))$	Nec 8.
10.	$(\Box\sim\Diamond Q \rightarrow \Box\sim(\Diamond P \wedge \Diamond Q))$	K 9
11.	$(\Diamond(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond \Diamond Q)$	TC 10, Def. of “$\Diamond$”
12.	$(\Diamond \Diamond Q \rightarrow \Diamond Q)$	Dual of (4)
13.	$(\Diamond(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond Q)$	TC 11, 12
14.	$((\Diamond P \wedge \Diamond Q) \leftrightarrow \Diamond(\Diamond P \wedge \Diamond Q))$	TC1, 7, 13

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In KT5. Every instance of (4) is provable in KT5.

7. True or false? Explain your answer: A disjunction $(\Box\varphi \vee \Box\psi)$ is in S4 if and only is either $\Box\varphi$ is in S4 or $\Box\psi$ is in S4.

True. The right-to-left direction is trivial. $(\Box\varphi \vee \Box\psi)$ is a tautological consequence of $\Box\varphi$ and of $\Box\psi$. To get the left-to-right, suppose that neither $\Box\varphi$ nor $\Box\psi$ is in S4. Then there are transitive, reflexive models $<W_1,R_1,I_1,@_1>$ and $<W_2,R_2,I_2,@_2>$ of $\sim\Box\varphi$ and $\sim\Box\psi$, respectively, where we may choose the models so that W_1 and W_2 are disjoint. Pick something not an element of either W_1 or W_2 to play the role of the actual world @, and let $W = W_1 \cup W_2 \cup \{@\}$. Let $R = R_1 \cup R_2 \cup (\{@\} \times W)$, so that R is reflexive and transitive and @ has access to every world. Define an intepreteation function I by stipulating, for α atomic:

$$\begin{aligned} I(\alpha,w) &= I_1(\alpha,w) \text{ if } w \in W_1; \\ &= I_2(\alpha,w) \text{ if } w \in W_2; \text{ and} \\ &= T \text{ if } w = @. \text{ (It doesn't matter what the say for rhe third case.)} \end{aligned}$$

An easy induction shows that, for any world $w \in W_1$ and formula θ, θ is true in $<W_1,R_1,I_1,w>$ iff θ is true in $<W,R,I,w>$. Since $\Box\varphi$ is false in $<W_1,R_1,I_1,@_1>$ there is a world v with $@_1 R_1 v$ and φ false in $<W_1,R_1,I_1,v>$. φ is false in $<W,R,I,v>$. Since @Rv, $\Box\varphi$ is false in $<W,R,I,@>$. Exactly the same argument shows that $\Box\psi$ is false in $<W,R,I,@>$. Thus $(\Box\varphi \vee \Box\psi)$, being false in $<W,R,I,@>$, is not in S4.

8. True or false? Explain your answer: A disjunction $(\Box\varphi \vee \Box\psi)$ is in S5 if and only is either $\Box\varphi$ is in S5 or $\Box\psi$ is in S5.

False. “$(\Box\Diamond P \vee \Box\Diamond\sim P)$” is in S5, but neither disjunct is.