Subject 24.244. Modal Logic. Answers to the fourth p-set.

1. Does "$(P \rightarrow Q)$" intuitionistically imply "$(\sim P \vee Q)$" or *vice versa*? Explain your answer.
"$(P \rightarrow Q)$" doesn't intuitionistically imply "$(\sim P \vee Q)$."
Take a model with two worlds, @ and w, with R = {<@,@>, <@,w>, <w,w>} and with both "P" and "Q" true in w only. "$(P \rightarrow Q)$" is true in @ but "$(P \rightarrow Q)$" is not.

"$(\sim P \vee Q)$" intuitionistically implies "$(P \rightarrow Q)$."

1.	$\{\sim P, P\} \vdash_{Int} \bot$	Law of contradiction
2.	$\{\bot\} \vdash_{Int} Q$	Ex contradictions quodlibet
3.	$\{\sim P, P\} \vdash_{Int} Q$	Transitivity 1, 2
4.	$\{\sim P\} \vdash_{Int} (P \rightarrow Q)$	Conditional proof 3
5.	$\{Q, P\} \vdash_{Int} Q$	Identity
6.	$\{Q\} \vdash_{Int} (P \rightarrow Q)$	Conditional proof 5
7.	$\{(\sim P \vee Q)\} \vdash_{Int} (P \rightarrow Q)$	Proof by cases 4, 6

2. Does "$(P \rightarrow Q)$" intuitionistically imply "$\sim (P \wedge \sim Q)$" or *vice versa*? Explain your answer.
"$(P \rightarrow Q)$" intuitionistically implies "$\sim (P \wedge \sim Q)$."

1.	$\{(P \wedge \sim Q)\} \vdash_{Int} P$	"$\wedge$"-elimination
2.	$\{(P \rightarrow Q), P\} \vdash_{Int} Q$	Modus ponens
3.	$\{(P \rightarrow Q), (P \wedge \sim Q)\} \vdash_{Int} Q$	Transitivity 1, 2
4.	$\{(P \wedge \sim Q)\} \vdash_{Int} \sim Q$	"$\wedge$"-elimination
5.	$\{Q, \sim Q\} \vdash_{Int} \bot$	Law of contradiction
6.	$\{(P \wedge \sim Q), (P \rightarrow Q)\} \vdash_{Int} \bot$	Identity, transitivity 2, 4, 5
7.	$\{(P \rightarrow Q)\} \vdash_{Int} \sim (P \wedge \sim Q)$	Intuitionistic reductio 6

"$\sim(P \wedge \sim Q)$" does not intuitionistically imply "$(P \rightarrow Q)$."
Take a model with two world, @ and w, with R = {<@,@>, <w,w>, <@,w>} and with "P" true in both world and "Q" true in w only.

3. Does "$(P \rightarrow \bot)$" intuitionistically imply "$\sim P$" or *vice versa*? Explain your answer.

"$(P \rightarrow \bot)$" intuitionistically implies "$\sim P$."

1.	$\{(P \rightarrow \bot), P\} \vdash_{Int} \bot$	Modus ponens
2.	$\{(P \rightarrow \bot)\} \vdash_{Int} \sim P$	Intuitionisitic reduction 1

"$\sim P$" intuitionistically implies "$(P \rightarrow \bot)$."

1.	$\{\sim P, P\} \vdash_{Int} \bot$	Law of contradiction
2.	$\{\sim P\} \vdash_{Int} (P \rightarrow Q)$	Conditional proof, 1

4. Show that Peirce's law, "$(((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi)$," is valid classically but not intuitionistically.
It's valid classically:

1.	$\{\sim \varphi, \varphi\} \vdash_{Class} \bot$	Law of contradiction

2. $\{\perp\} \vdash_{claaa}$ Ex contradictione quodlibet
3. $\{\sim\varphi, \varphi\} \vdash_{class} \psi$ Transitivity 1, 2
4. $\{\sim\varphi\} \vdash_{Class} (\varphi \rightarrow \psi)$ Conditional proof 3
5. $\{((\varphi \rightarrow \psi) \rightarrow \varphi), (\varphi \rightarrow \psi) \vdash_{Class} \varphi$ Modus ponens
6. $\{((\varphi \rightarrow \psi) \rightarrow \varphi), \sim\varphi\} \vdash_{Class} \varphi$ Transitivity 4, 5
7. $\{((\varphi \rightarrow \psi) \rightarrow \varphi), \sim\varphi\} \vdash_{Class} \sim\varphi$ Identity
8. $\{((\varphi \rightarrow \psi) \rightarrow \varphi), \sim\varphi\} \vdash_{Class} \perp$ Law of contradiction and transitivity 6, 7
9. $\{((\varphi \rightarrow \psi) \rightarrow \varphi)\} \vdash_{Class} \sim\sim\varphi$ Intuitionistic reduction 8
10. $\{\sim\sim\varphi\} \vdash_{Class} \varphi$ Double negation elimination
11. $\{((\varphi \rightarrow \psi) \rightarrow \varphi)\} \vdash_{Class} \varphi$ Transitivity 9, 10
12. $\varnothing \vdash_{Class} (((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi)$ Conditional proof 11

It's not valid intuitionistically:
Take a model with two worlds, @ and w, with R = {<@,@>, <@,w>, <w,w>} and with φ true in w only and ψ true in neither world. (φ → ψ) isn't true isn't true in either world. So ((φ → ψ) → φ) is true in both world, even though φ isn't true in @.

5. Which of the following versions of de Morgan's law are valid intuitionistically? Explain"

a) $(\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi))$. [$(\varphi \leftrightarrow \psi)$ abbreviates $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.]
Invalid. The right-to-left direction is valid, but the left-to-right is not. Make a model with three worlds, @, w, and v and R = {<@,@>,<w,w>,<v,v>,<@,w>,<@,v>}. Let φ be true in w only and ψ true in v only. Then neither ~ φ nor ~ ψ is true in @, but ~ (φ ∧ ψ).

b) $(\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi))$.
Valid.
1. $\{\varphi\} \vdash_{Int} (\varphi \vee \psi)$ "∨"-introduction
2. $\{\varphi, \sim(\varphi \vee \psi)\} \vdash_{Int} (\varphi \vee \psi)$ Transitivity 1
3. $\{\varphi, \sim(\varphi \vee \psi)\} \vdash_{Int} \sim(\varphi \vee \psi)$ Identity
4. $\{\varphi, \sim(\varphi \vee \psi)\} \vdash_{Int} \perp$ Law of contradiction, transitivity 2, 3
5. $\{\sim(\varphi \vee \psi)\} \vdash_{Int} \sim\varphi$ Intuitionistic reductio 4
6 $\{\psi\} \vdash_{Int} (\varphi \vee \psi)$ "∨"-introduction
7. $\{\psi, \sim(\varphi \vee \psi)\} \vdash_{Int} (\varphi \vee \psi)$ Transitivity 6
8. $\{\psi, \sim(\varphi \vee \psi)\} \vdash_{Int} \sim(\varphi \vee \psi)$ Identity
9. $\{\psi, \sim(\varphi \vee \psi)\} \vdash_{Int} \perp$ Law of contradiction, transitivity 7, 8
10. $\{\sim(\varphi \vee \psi)\} \vdash_{Int} \sim\psi$ Intuitionistic reductio 9
11. $\{\sim\varphi, \sim\psi\} \vdash_{Int} (\sim\varphi \wedge \sim\psi)$ "∧'-introduction
12. $\{\sim(\varphi \vee \psi)\} \vdash_{Int} (\sim\varphi \wedge \sim\psi)$ Transitivity 5, 10, 11
13. $\varnothing \vdash_{Int} (\sim(\varphi \vee \psi) \rightarrow (\sim\varphi \wedge \sim\psi))$ Conditional proof 12
14. $\{\varphi, \sim\varphi, \sim\psi\} \vdash_{Int} \perp$ Law of contradiction, identity, transitivity
15. $\{\varphi, \sim\varphi, \sim\psi\} \vdash_{Int} \perp$ Law of contradiction, identity, transitivity
16. $\{(\varphi \vee \psi), \sim\varphi, \sim\psi\} \vdash_{Int} \perp$ "∨"-introduction 14, 15
17. $\{\sim\varphi, \sim\psi\} \vdash_{Int} \sim(\varphi \vee \psi)$ Intuitionistic reductio 16

18. $\{(\sim \varphi \wedge \sim \psi)\} \vdash_{Int} \sim \varphi$ "∧"-elimination
19. $\{(\sim \varphi \wedge \sim \psi)\} \vdash_{Int} \sim \psi$ "∧"-elimination
20. $\{(\sim \varphi \wedge \sim \psi)\} \vdash_{Int} \sim (\varphi \vee \psi)$ Transitivity 17, 18, 19
21. $\varnothing \vdash_{Int} ((\sim \varphi \wedge \sim \psi) \rightarrow \sim (\varphi \vee \psi))$ Conditional proof 20
22. $\varnothing \vdash_{Int} \sim(\varphi \vee \psi) \leftrightarrow (\sim \varphi \wedge \sim \psi))$ "∧"-introduction, transitivity, def. of "↔" 13, 21

6. **An intuitionistic model is a transitive, reflexive Kripke model that meets the condition that, whenever v is accessible from w, any atomic sentence true in w is true in v. Give a set of axioms for the set of formulas of the modal sentential calculus true in every intuitionistic model, and sketch a proof that your axiom system is sound and complete.**
The axioms will be the instances of schemata (K), (T), (4), together will all sentences $(\varphi \supset \Box\varphi)$, for φ atomic. The rules are TC and Nec. The axioms are true in every world in every intuitionistic model, and the set of sentences true in every world in every intuitionist model is closed under TC and Nec. Because the formulas derivable from the axioms by the rules constitute a normal modal syttem that includes (T) and (4), its canonical frame is transitive and reflexive, and it enjoys the property that any atomic formula true in a world is true in every world accessible from that world. So the canonical frame is an intuitionistic frame. So for any formula that isn't derivable from the axioms, there will be a world that excludes the formula, and that world will be a world in an intuitionistic model in which the formula is false.

7. **We showed that a formula φ is an intuitionistic consequence of a set of formulas Γ if and only if the Gödel translation of φ is an S4-consequence of the Gödel translations of the members of Γ. Sketch a proof that φ is a classical consequence of Γ if and only if the Gödel translation of φ is an S5-consequence of the Gödel translations of the members of Γ.**
(⇒) A sentence is a classical consequence of a set of premises iff it's derivable from the premises by the intuitionistic rules supplemented with DNE. Adding DNE to the rules gives the same outcomes as taking the conditionals $(\sim \sim \psi \supset \psi)$ as axioms. Only finitely many of these axioms appear in a derivation, so if θ is derivable from Γ, then there are formulas $\psi_1, \psi_2,..., \psi_n$ such that θ is derivable intuitionistically from $\Gamma \cup \{(\sim \sim\psi_i \supset \psi_i): 1 \leq i \leq n\}$. Gödel's result tells us that tr(θ) is derivable in S4 from $\{tr(\gamma): \gamma \in \Gamma\} \cup \{tr(\sim \sim \psi_i \supset \psi_i): 1 \leq i \leq n\}$. $tr(\sim \sim \psi_i \supset \psi_i)$ is equal to $\Box(\Box{\sim}\Box{\sim}tr(\psi_i) \supset tr(\psi_i))$. Since $tr(\psi_i)$ begins with a "□," this formula will be a theorem of S5, so tr(θ) will be an S5 consequence of the image under tr of Γ.

(⇐) If χ isn't a classical consequence of Γ, then there is a complete story @ that includes Γ and excludes χ. Define a model <W,R,I,@> by stipulating that W = {@}, R = {<@,@>}, and I(@.φ) = T iff $\varphi \in$ @. Because there is only the one world, a formula θ is true at @ if and only if $\Box\theta$ is true in @. So, for any sentential calculus formula ψ, ψ is true in @ if an only if tr(ψ) is true in @. So <W,R,I,@> is an S5 model in which tr(χ) is false and the members of tr "$\Gamma$are ll true.

8. **True or false? Explain your answer: A formula χ is a classical consequence of a set of formulas Γ if and only if $\sim \sim\varphi$ is a intuitionistic consequence of Γ.**
(⇐) If $\sim \sim \varphi$ is derivable intuitionistically from the empty set, it's derivable classically from the empty set, so it's classically valid. $(\sim \sim\varphi \supset \varphi)$ is classically valid, so φ is classically valid.

(⇒) If $\sim \sim\varphi$ isn't derivable from the empty set intuitionistically, then, by going through the sentences one by one, we can form a maximal set Ω from which $\sim \sim \varphi$ isn't intuitionistically derivable. I claim that Ω is a complete story. If so, then it's a complete story that doesn't include $\sim\sim\varphi$, so it doesn't include φ, so φ isn't classically valid.

Any formula derivable intuitionistically from Ω is in Ω. Therefore, a conjunction in in Ω iff both its conjuncts are.

For the same reason, a disjunction will be in Ω if both its disjuncts are. For the converse, suppose that neither ψ nor θ is in Ω; Then $\sim\sim\varphi$ is intuitionistically derivable from $\Omega \cup \{\psi\}$ and from $\Omega \cup \{\theta\}$. By proof by cases, $\sim\sim\varphi$ is intuitionistically derivable from $\Omega \cup \{(\psi \vee \theta)\}$. So $(\psi \vee \theta)$ isn't in Ω.

In showing that the clauses for conditionals and negations in the definition of complete story are satisfied by Ω, the key observation is that $\sim\varphi$ is in Ω. If $\sim\varphi$ weren't in Ω, then $\sim\sim\varphi$ would be intuitionistically derivable from $\Omega \cup \{\sim\varphi\}$, and so, by intuitionistic reducio, $\sim\sim\varphi$ would be intuitionistically derivable from Ω

For conditionals, we need to show that $(\psi \supset \theta)$ is in Ω iff ψ isn't in Ω or θ is. If $(\psi \supset \theta)$ is in Ω and ψ is in Ω, θ is in Ω by modus ponens. If ψ isn't in Ω, $\sim\sim\varphi$ is intuitionistically derivable from $\Omega \cup \{\psi\}$. Since $\sim\varphi$ is in Ω, *ex contradictione quodlibet* (ECQ) tells us that θ is derivable from $\Omega \cup \{\psi\}$ By conditional proof, $(\psi \supset \theta)$ is intuitionistically derivable from Ω, and so an element of Ω. Finally, if θ in in Ω, θ is in $\Omega \cup \{\psi\}$, so θ is intuitionistically derivable from $\Omega \cup \{\psi\}$, so $(\psi \supset \theta)$ is intuitionistically derivable from Ω, by conditional proof, so $(\psi \supset \theta)$ is an element of Ω.

ψ and $\sim\psi$ aren't both in Ω, because $\{\psi, \sim\psi\}$ intuitionistically entails $\sim\sim\varphi$, by ECQ. If ψ isn't in Ω, $\Omega \cup \{\psi\}$ intuitionistically entails $\sim\sim\varphi$. Since $\sim\varphi$ is in Ω, $\Omega \cup \{\psi\}$ intuitionistically entails $\perp$, by ECQ. By intuitionistic reduction, Ω intuitionistically entails $\sim\psi$, so it's in Ω..