**Subject 24.244. Modal logic. Answers to the fifth p-set.**

1. **Describe a frame <W,R,I> with the properties that every formula in KT is true in every world in the frame and that R isn't reflexive, and explain why it has those properties; or else, explain why there is no such frame.**

   Take a frame with two worlds, w and v, neither of which has access to itself and each of which has access to the other. Let the same atomic sentences be true in each world. An induction on the complexity of formulas shows that all the same sentences are true in each world. Moreover, $\Box\varphi$ is true in w iff $\varphi$ is true in v iff $\varphi$ is true in v, and $\Box\varphi$ is true in v iff $\varphi$ is true in w iff $\varphi$ is true in v. So all the instances of (T) are true in every world in the frame. The sentences true in every world in the frame is a normal modal system. So includes KT

2. **Describe a frame <W,R,I> with the properties that every formula in K4 is true in every world in the frame and that R isn't transitive, and explain why it has those properties; or else, explain why there is no such frame.**

   Use the same frame as in problem 1. It's not reflexive, because we have wRv and vRw but not wRw. If $\Box\,\varphi$ is true in w, $\varphi$ is true in v, so $\varphi$ is true in w, so $\Box\,\varphi$ is true in v, so $\Box\,\Box\,\varphi$ is true in w. So $(\Box\,\varphi \rightarrow \Box\,\Box\varphi)$ is true in w, for every sentence $\varphi$. A symmetric argument shows that all the instances of (4) are true in v. So the set of sentences true in both worlds is a normal modal system that includes (4).

3. **Which of the following formulas are in GL? Explain your answers:**

   **a) $((\Diamond P \wedge \Diamond Q) \rightarrow \Diamond(\Diamond P \wedge \Diamond Q))$.**

   **Not in GL.** Take a model with two worlds @ and w, with R= {<@,w>} and with "P" and "Q" true only in w. Then "$(\Diamond P \wedge \Diamond Q)$" is true in @ only, and "$\Diamond(\Diamond P \wedge \Diamond Q)$" isn't true in either world.

   **b) $(\Diamond(\Diamond P \wedge \Diamond Q) \rightarrow (\Diamond P \wedge \Diamond Q))$.**

   **In GL.**

| | | |
|---|---|---|
| 1. | $\sim\Diamond P \rightarrow \sim(\Diamond P \wedge \Diamond Q)$ | (Taut) |
| 2. | $\Box(\sim\Diamond P \rightarrow \sim(\Diamond P \wedge \Diamond Q))$ | Nec 1 |
| 3 | $(\Box{\sim}\Diamond P \rightarrow \Box{\sim}(\Diamond P \wedge \Diamond Q))$ | K 2 |
| 4. | $\Diamond{\sim}(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond P$ | TC 1, Def. of "$\Diamond$" |
| 5. | $\sim\Diamond Q \rightarrow \sim(\Diamond P \wedge \Diamond Q)$ | (Taut) |
| 6. | $\Box(\sim\Diamond Q \rightarrow \sim(\Diamond P \wedge \Diamond Q))$ | Nec 5 |
| 7 | $(\Box{\sim}\Diamond Q \rightarrow \Box{\sim}(\Diamond P \wedge \Diamond Q))$ | K 6 |
| 8. | $\Diamond{\sim}(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond Q$ | TC 7, Def. of "$\Diamond$" |
| 9. | $(\Diamond(\Diamond P \wedge \Diamond Q) \rightarrow (\Diamond P \wedge \Diamond Q))$ | TC 4, 8 |

   **c) $(\Diamond(P \leftrightarrow Q) \rightarrow \Box\Diamond(P \leftrightarrow Q))$.**

   **Not in GL.** Use the same model as 3a). "$\Diamond(P \leftrightarrow Q)$" is true in @ only. Since it's not true in w, "$\Box\Diamond(P \leftrightarrow Q)$" isn't true in @.

   **d) $(\Diamond(\Diamond P \wedge \Diamond Q) \leftrightarrow (\Diamond\Diamond P \wedge \Diamond\Diamond Q))$.**

   **Not in GL.** The left-to-right conditional is in GL. In fact, it's in K. But not the right-to-left. Take a model with five worlds, @, t, u, v, and w, and with R = {<@,t>, <@,u>, <@,v>, <@,w>, <t,u>, <v,w>}. Let "P" be true in u only and "Q" true in w only. "$\Diamond P$" is true in t, so "$\Diamond\,\Diamond P$" is true in @. “$\Diamond$Q” is true in w, so “$\Diamond\Diamond$Q” is true in @. “($\Diamond$P $\wedge$ $\Diamond$Q)” is true in @ only, so “$\Diamond$($\Diamond$P $\wedge$ $\Diamond$Q)” isn’t true in @.

**4. Show that, when we defined GL, including axiom schema (4) was redundant, so that GL is equal to KL, the smallest normal modal system that contains (L); this was discovered by Dick de Jongh. [Hint: The relevant instance of (L) is $(\Box(\Box(\varphi \wedge \Box \varphi) \rightarrow (\varphi \wedge \Box \varphi)) \rightarrow \Box(\varphi \wedge \Box \varphi))$.]**

| | | |
|---|---|---|
| 1. | $((\varphi \wedge \Box\varphi) \rightarrow \varphi)$ | (Taut) |
| 2. | $\Box((\varphi \wedge \Box \varphi) \rightarrow \varphi)$ | Nec 1 |
| 3. | $(\Box(\varphi \wedge \Box\varphi) \rightarrow \Box \varphi)$ | K 2 |
| 4. | $(\varphi \rightarrow (\Box(\varphi \wedge \Box \varphi) \rightarrow (\varphi \wedge \Box\varphi)))$ | TC 3 |
| 5. | $\Box(\varphi \rightarrow (\Box(\varphi \wedge \Box\varphi) \rightarrow (\varphi \wedge \Box\varphi)))$ | Nec 4 |
| 6. | $(\Box\varphi \rightarrow \Box(\Box(\varphi \wedge \Box\varphi) \rightarrow ( \varphi \wedge \Box\varphi)))$ | K 5 |
| 7. | $(\Box(\Box(\varphi \wedge \Box\varphi) \rightarrow (\varphi \wedge \Box\varphi)) \rightarrow \Box(\varphi \wedge \Box\varphi))$ | (L) |
| 8. | $(\Box\varphi \rightarrow \Box(\varphi \wedge \Box \varphi))$ | TC 6,7 |
| 9. | $((\varphi \wedge \Box\varphi) \rightarrow \Box\varphi)$ | (Taut) |
| 10. | $\Box((\varphi \wedge \Box\varphi) \rightarrow \Box\varphi)$ | Nec 9 |
| 11. | $(\Box(\varphi \wedge \Box \varphi) \rightarrow \Box \Box\varphi)$ | K 19 |
| 12. | $(\Box\varphi \rightarrow \Box \Box\varphi)$ | TC 8, 11 |

**5. Take a sentence α so that α is provably (in PA) equivalent to $\neg \mathrm{Bew}_{PA}([\ulcorner \neg \alpha \urcorner])$. Is α decidable in PA? Is it true (in the standard model)? Explain your answers.**

$(\mathrm{Bew}_{PA}([\sim \alpha]) \rightarrow \sim \alpha)$ is provable, so by Löb’s theorem, $\sim \alpha$ is true in provable. So α is decidable and false.

**6. Take a sentence β so that β is provably equivalent to $(\mathrm{Bew}_{PA}([\ulcorner \beta \urcorner]) \vee \sim \mathrm{Bew}_{PA}([\ulcorner \beta \urcorner]))$. Is β decidable in PA? Is it true? Explain your answer**.

β is provably equivalent to a tautology, so it’s true and decidable.

**7. Take a sentence δ so that δ is provably equivalent to $(\mathrm{Bew}_{PA}([\ulcorner\delta \urcorner]) \wedge \mathrm{Bew}_{PA}([\ulcorner \neg \delta \urcorner)])$. Is δ decidable in PA? Is it true?**

$(\delta \wedge \sim \delta)$ is provably equivalent to “$\sim 0=0$,” so $(\mathrm{Bew}_{PA}([\ulcorner \delta \urcorner) \wedge \mathrm{Bew}_{PA}([\ulcorner \sim \delta \urcorner]))$ is provably equivalent to $\mathrm{Bew}_{PA}([\ulcorner\sim 0=0 \urcorner])$, which is tautologically equivalent of $\sim$ CON(PA). So δ is provably equivalent to $\sim$ CON(PA), which is false, but by the second incompleteness theorem, not refutable.

**8. Show that every sentence of the form $\Diamond \Box \varphi$ is in GLS.**
**Direct proof:**

The following formulas are in GL:

| | | |
|---|---|---|
| 1. | $(\sim \Box \varphi \rightarrow (\Box\varphi \rightarrow \varphi))$ | (Taut) |
| 2. | $\Box(\sim \Box \varphi \rightarrow (\Box\varphi \rightarrow \varphi))$ | Nec 1 |
| 3. | $(\Box{\sim}\Box\varphi \rightarrow \Box(\Box\varphi \rightarrow \varphi))$ | K 2 |
| 4. | $(\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi)$ | (L) |
| 5. | $(\Box{\sim}\Box\varphi \rightarrow \Box\varphi)$ | TC 3, 4 |
| 6. | $(\sim \Box \varphi \rightarrow \Diamond\Box\varphi)$ | TC 5, Def. of “$\Diamond$” |

So the following formulas are in GLS:

| | | |
|---|---|---|
| 1. | $(\sim \Box\varphi \rightarrow \Diamond\Box\varphi)$ | GL |
| 2. | $(\Box\varphi \rightarrow \Diamond\Box\varphi)$ | Dual of (T) |
| 3. | $\Diamond \Box \varphi$ | TC, 1, 2 |

**Indirect proof:** We know from the second incompleteness theorem that we can't prove that we can't prove φ. So $\sim \Box \sim \Box\varphi$ is always true, hence a member of GLS.