Stalnaker Semantics

Robert Stalnaker[1] revolutionized the study of conditionals by applying to it the methods of possible-worlds semantics. He regarded a conditional as true in a world w iff its consequent is true in the world most similar to w in which the antecedent is true.

Define a *candidate Stalnaker model* to be a quadruple $\langle W,I,f,@\rangle$, where W is a nonempty set of things we're calling worlds, I is a function {atomic formulas} × W → {T,F}, f is a partial function[2]: W × {formulas} → W, and @, the actual world of the model, is an element of W. We define what it is for a formula to be true in a world in the model:

An atomic formula α is true in w iff I(α,w) = T
A conjunction is true in w iff both conjuncts are true in w.
A disjunction is true in w iff one or both disjuncts are true in w.
A negation is true in w iff the negatum isn't true in w.
"_|_" isn't true in w.
A conditional $(\varphi > \psi)$ is true in w iff either $\langle w,\varphi\rangle$ isn't in Dom(f) or $\langle w,\varphi\rangle$ is in Dom(f) and ψ is true in f(w,φ).

A formula is true in the model iff it's true in @.

A *Stalnaker model* is a candidate Stalnaker model that meets these further conditions:

φ is true in f(w,φ).
If φ is true in w, f(w,φ) = w.
If ψ is true in f(w,φ), $\langle w,\psi\rangle$ is in Dom(f).
If ψ is true in f(w,φ) and φ is true in f(w,ψ), f(w,φ) = f(w,ψ).

A formula is a *Stalnaker theorem* iff it's derived from the following axioms by the following rules:

Axioms:
(Conditional K) $((\varphi > (\psi \rightarrow \theta)) \rightarrow ((\varphi > \psi) \rightarrow (\varphi > \theta)))$
(Reflexive law) $(\varphi > \varphi)$
(Modus ponens) $(((\varphi > \psi) \wedge \varphi) \rightarrow \psi)$
(Centering) $((\varphi \wedge \psi) \rightarrow (\varphi > \psi))$
(Equivalent antecedents) $(((\varphi > \psi) \wedge (\psi > \varphi)) \supset ((\varphi > \theta) \leftrightarrow (\psi > \theta)))$
(Conditional excluded middle) $((\varphi \Rightarrow \psi) \vee (\varphi \Rightarrow \sim \psi))$

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[1] *A Theory of Conditionals* (Blackwell, 1968).

[2] That is, s is a function whose domain is a subset of W × {sentences} and whose range is a subset of W.

Rules:
(Tautological consequence) You may write any tautology or tautological consequence of things you've written earlier.
(Conditionalization). If you've written ψ, you may write (φ ⇒ ψ).

A f$_{orm}$ula is a *Stalnaker consequence* of a set of sentences Γ iff it's a tautological consequences of {Stalnaker theorems} ∪ Γ. Γ is *Stalnaker-consistent* iff Γ ∪ {Stalnaker theorems} is tautologically consistent. By examining the axioms and rules we verify:

Soundness Theorem. Every Stalnaker theorem is true in every Stalnaker model.

Completeness Theorem. Every formula that is true in every Stalnaker model is a Stalnaker theorem.

Sketch of proof: Suppose that χ isn't a Stalnaker theorem. Thene we can form a maximal Stalnaker-consistent set @ of formulas that doesn't have χ as a Stalnaker consequence. L@ is a complete story. Form a candidate Stalnaker mode <W,I,f,@> by lettingW be the set of maximal Stalnaker-consistent sets of formulas. Let I(α,w) = T iff α ∈ w. <w,φ> is in Dom(f) iff (φ > _|_) isn't in w. If <w,φ> is in Dom(f), f(w,φ) = {ψ: (φ > ψ) ∈ w}. One can verify, although it takes a while, that, for any formula φ, φ is true in a world if and only if it's an element of the world and that <W,I,f,@> is a Stalnaker model.⊠

It follows that a set of formulas is Stalnaker consistent iff there is a Stalnaker model in which its members are all true. φ is a Stalnaker consequence of Γ iff φ is true in every Stalnaker model in which all the members of Γ are true. We have compactness.

There is a modal logic in the background. If φ is possible in w, then there is a world accessible from w in which φ is true, and so, on Stalnaker's assumptions, there is a closest world to w among the worlds accessible from w in which φ is true. So f(w,φ) is defined. Following up on this idea, we can introduce modal operators as defined symbols of the modal language thus:

$\Diamond\ \varphi =_{\text{Def}} \sim (\varphi > \_|\_)$.

$\Box\ \varphi =_{\text{Def}} (\sim \varphi > \_|\_0$.

This gives us KT as the modal logic. The other familiar modal principles can be adopted as additional axiom.