Montague's Theorem

Robinson's Q:

$(\forall x) \sim Sx = 0.$
$(\forall x)(\forall y)(Sx = Sy \rightarrow x = y)$
$(\forall x)(x+0) = x.$
$(\forall x)(\forall y)(x + Sy) = S(x + y).$
$(\forall x)(x\cdot 0 = 0.$
$(\forall x)(\forall y)(x \cdot Sy) = ((x \cdot y) + x)$
$(\forall x)(x \leq 0 \leftrightarrow x = 0)$
$(\forall x)(\forall y)(x \leq Sy \leftrightarrow (x \leq y \vee x = Sy)).$
$(\forall x)(\forall y)(x \leq y \vee y \leq x).$

Q is much weaker that PA. For instance, it doesn't prove the commutative law of addition. Nevertheless, it is strong enough to prove the Self-reference Lemma.

Montague's Theorem. In the language obtained from the language of arithmetic by addin a new predicate "Nec" to represent necessity, there isn't any consistent set of sentences that:

(i) contains the logical consequences of Q;
(ii) contains all sentences of the form $(\mathrm{Nec}([\ulcorner \varphi \rightarrow \psi \urcorner]) \rightarrow (\mathrm{Nec}([\ulcorner \varphi \urcorner]) \rightarrow \mathrm{Nec}([\ulcorner \psi \urcorner]))$;
(iii) contains all sentences of the form $(\mathrm{Nec}([\ulcorner \varphi \urcorner]) \rightarrow \varphi)$;
(iv) contains $\mathrm{Nec}([\ulcorner \varphi \urcorner])$ whenever it contains φ; and
(v) is closed under tautological consequence.

Proof: Suppose Γ is such a set. The Self-reference Lemma gives us a sentence ν such that $(\nu \leftrightarrow \sim \mathrm{Nec}([\ulcorner \nu \urcorner]))$ is a consequence of Q. Consequently the following are in Γ:

1.	$(\sim \mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu)$	By (i) and (v)
2.	$(\mathrm{Nec}([\ulcorner \nu \urcorner] \rightarrow \nu)$	By (iii)
3.	ν	From 1 and 2 by (v)
4.	$\mathrm{Nec}([\ulcorner \nu \urcorner])$	From 3 by (iv)
5.	$(\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \sim \nu)$	From (i)
6.	$\sim \nu$	From 4 and 5 by (v)

Montague thought that any system of modal logic worth the name must include KT, so he concluded that treating necessity as a property of sentences expressed by a predicate "Nec" would lead inevitable to paradoxes. He concluded that, rather than express necessity by a predicate true of the necessary sentences, we should express necessity with a modal operator "□."