Subject 24.244. Answers to the sixth p-set

Hamblin's 15 tense theorem tells us about sequences of tense operators starting with "H" ("it has always been the case that"), "P" ("it has sometimes been the case that"), "G" ("it will always be the case tbat"), and "F" ("it will sometimes be the case that"). He gave a diagram of 15 prefixes (including the null prefix). He showed that one of the 15 prefixes implies another if and only is they are connected by a path of arrows, and that any prefix optained by prefixing one of the operators to a prefix in the diagram is eqivalent to a prefix in the diagram. Here, to say that φ implies ψ means that (φ → ψ) is derivable within the following system of axioms, which assume that time is totally ordered, dense, and continuous, without beginning or end.

Axioms. We use "Pp" to abbreviate "~H~p" and "Fp" to abbreviate "~G~p."

(i)	**$(G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq))$**
(ii)	**$(p \rightarrow GPp)$**
(iii)	**$(Gp \rightarrow GGp)$**
(iv)	**$((Fp \wedge Fq) \rightarrow (F(p \wedge Fp) \vee (F(p \wedge q) \vee F(Fp \vee q))))$**
(v)	**$(Gp \rightarrow Fp)$**
(vi)	**$(Fp \rightarrow FFp)$**
(vii)	**$((Fp \wedge FG{\sim}p) \rightarrow F(HFp \wedge G{\sim}p))$**

Rules.
Tautological consequence.
Temporal generalization: From φ to infer Gφ and Hφ.
Mirror image rule: From a formula, you may derive the correspoinding
formula obtained by exchanging "G" and "H" and also "F" and "P."
In giving the derivations, I will skip over some small steps, like eliminating double negation and exchanging "~P" and "H" inside a formula. Writing out all the details in unbearably tedious. The proof that, for normal modal systems, the substitution rule is conservative (i.e., everything you can prove using the rule you can prove without it) carries over to tense logic,so we know that the details cn be filled in.

1. Show, by giving a derivation, that "PGp" implies "GPp."

1.	$({\sim}p \rightarrow GP{\sim}p)$	(ii), substituting "~ p" for "p"
2.	$({\sim}p \rightarrow G{\sim}H{\sim}{\sim}p)$	1, def. of "P"

3.	$(\sim p \rightarrow \sim \sim G \sim Hp)$	Double negation intro. and elim. 2
4.	$(\sim p \rightarrow \sim FHp)$	Def. of "G"
5.	$(FHp \rightarrow p)$	TC 4
6.	$(PGp \rightarrow p)$	Mirror 5
7.	$(p \rightarrow GPp)$	(ii)
8.	$(PGp \rightarrow GPp)$	TC 6, 7

2. Show, by giving a derivation, that "PGp" implies "HFp."

1.	$(PGp \rightarrow p)$	Line 6 of problem 1
2.	$(p \rightarrow GPp)$	(ii)
3.	$(p \rightarrow HFp)$	Mirror 2
4.	$(PGp \rightarrow HFp)$	TC 1, 3

3. Show, by giving a derivation, that "PGp" implies "GFp."

1.	$(PGGp \rightarrow Gp)$	Line 6 of problem 1, substituting "Gp" for"p"
2.	$(Gp \rightarrow GGp)$	(iii)
3.	$\sim GGp \rightarrow \sim Gp)$	TC 2
4.	$H(\sim GGp \rightarrow \sim Gp)$	Temporal generalization 3
5.	$(H\sim GGp \rightarrow H\sim Gp)$	From 5 by (i) and TC
6.	$(PGp \rightarrow PGGp)$	TC 4, Def. of "P"
7.	$(PGp \rightarrow Gp)$	TC 1, 6
8.	$(Gp \rightarrow Fp)$	(v)
9.	$G(Gp \rightarrow Fp)$	Temporalization 8
10.	$(GGp \rightarrow GFp)$	From 9 by (i)
11.	$(Gp \rightarrow GGp)$	(iii)
12.	$(PGp \rightarrow GFp)$	TC 7, 10, 11

4. Show that "GFp" doesn't imply "PGp" by giving a set S of real numbers such that (assuming lesser numbers correspond to earlier times and 0 corresponds to the present), if "p" is true at the times in S and at no other times, "GFp" will now be true and "PGp" now false.

Let S be the set of positive integers. Since for every positive real number x there is a positive integer greater than x, "GFp" will be true. Since for any negative real number y that are real numbers greater than y that aren't positive integers, "PGp" will be false.

5. Show, as in problem 4, that "PGp" doesn't imply "HPp."

Let S be the real numbers ≥ -2. Since "Gp" is true at -2, "PGp" is true. Since "Pp" is false at -2, "HPp" is false.

6. Show, as in problem 4, that "FGp" doesn't imply "GPp."

Let S be the set of real numbers ≥ 1. Since "Gp" is true at1, "FGp" is true. Since "Pp" is false as ½, "GPp" is false.

7. **Show that each of the four formulas you obtain by prefixing one of the four operators to "PGp" is equivalent to a formula in the diagram. You don't need to do derivations. Just show that, assuming times are ordered like the real numbers, at whatever times "p" is true, any formula obtained by prefixing one of the four operators to "PGp" will be true at the same times as one of the formulas in the diagram.**

Because PPφ is equivalent to Pφ for every φ, "PPGp" is logically equivalent to "PGp."

"HPGp" is equivalent to "HGp." **(⇒)** If "HPp" is now true, then for every past time x there is a time y such that "p" is true at all times after y. It follows that is true at all times after y, so "Gp" is true in x. Since x was an arbitriary past time, "HGp" is now true.
(⇐) If "HPGp" is false, there is a past time t at which "PGp" is false. So at any time u earlier than t, "Gp" is false. Since "Gp" is false at u, it hasn't always been true, so "HGp" is false.

"FPGp" is equivalent to "FGp." **(⇒)** If "FPGp" is true, then there is a future time x such that there is a time y earlier than x after which "p" is always true. So "p" is true at all times after x. So "Gp" is true at x and "FGp" is true now.
(⇐) If "FPGp" is false, then for any future time t, "PGp" is false it t. If "PGp" were true now, it would be true at t. So "PGp" is false now.

"GPGp" is equivalent to "Gp." **(⇒)** If "Gp" is false, there is a future time t at which "p" is false. At any time prior to t, "Gp" is going go be false, so "PGp" is false at t. Thus "GPGp" is false now.
(⇐) For sentence φ, if φ is true now, GPφ will be true now. Substititue "Gp" for φ.

8. **Show that each of the four formulas you obtains by prefixing one of the four operators to "HPp" is equivalent to a formula on the diagram. You don't need to do derivations. Talk about times instead.**
For any time t, "HPp" is true in t iff for every time there is an earlier time at which "p" is true. For this it doesn't matter what t is true. If "HPp" is true at one time, it's true at all. Thus the following are equivalent:

For each time, there is an earlier time at which "p" is true.

"HPp" is true now.
"HPp" was true at some earlier time.
"HPp" was true at every earlier time.
"HPp" will be true at some future time.
"HPp" will be true at every future time.

Hence the following are equivalent:

"HPp" is true.
"PHPp" is true.
"HHPp" is true.
"FHPp" is true.
"GHPp" is true.

9. Show that each of the for formulas you obtains by prefixing one of the four operators to "FPp" is equivalent to a formula on the diagram. You don't need to do derivations.

For any time t, "FPp" will be true in t iff there is a time at which "p" is true. For this, it doesn't matter what t is. If "FPp" is true at one time, it's true at all. Hence the following are equivalent:

There is a time at which "p" is true.
"FPp" is true now.
"FPp" was true at some past time.
"FPp" was true at every past time.
"FPp" will be true a some future.
"FPp" will be true at every future time.

Consequently, "FPp," "PFPp," "HFPp," "FFPp," and "GFPp" are equivalent.