Axioms of predicate logic.

All sentences of the following forms are axioms:

($\forall$Dist)	$(\forall v_i)(\varphi \rightarrow \psi) \rightarrow ((\forall v_i)\varphi \rightarrow (\forall v_i)\psi)$
(US)	$(\forall v_i)\varphi(v_i) \rightarrow \varphi(c)$
(Vac)	$(\varphi \leftrightarrow (\forall v_i)\varphi)$, v_i not free in φ
($\exists$Def)	$(\exists v_i)\varphi(v_i) \leftrightarrow \sim (\forall v_i) \sim \varphi(v_i)$
(Ref=)	$c = c$
(Sub=)	$(c=d \rightarrow (\varphi(c) \leftrightarrow \varphi(d))$

Rules:

TC	
UG	If c doesn't occur in $\varphi(v_i)$, then from $\varphi(c)$ you may infer $(\forall v_i)\varphi(v_i)$.

A model $\mathfrak{A}$ consists of a nonempty set $|\mathfrak{A}|$ and a function assigning an element of $|\mathfrak{A}|$ to each individual constant and a set of n-tuples from $|\mathfrak{A}|$ to each n-place predicate, subject to the condition that $\mathfrak{A}$("=") = $\{<b,b>: b \in |\mathfrak{A}|\}$. We write $c^{\mathfrak{A}}$ for $\mathfrak{A}(c)$ and $R^{\mathfrak{A}}$ for $\mathfrak{A}(R)$.
A variable assignment assigns an element of $|\mathfrak{A}|$ to each variable.

σ satisfies $R(\tau_1,\tau_2,...,\tau_n)$ iff $<b_1,b_2,..,b_n> \in \mathfrak{A}(R)$, where $b_i = \mathfrak{A}(\tau_i)$ if τ_i is a constant, $\sigma(\tau_i)$ if τ_i is a variable.
σ satisfies a disjunction iff it satisfies one or both disjuncts, a conjunction iff it satisfies both conjuncts, and a negation iff it fails to satisfy the negatum.
A v_i-variant of σ is a variable assignment that agrees with σ except possibly in the value it assigns to v_i.
σ satisfies $(\forall v_i)\varphi$ iff every v_i-variant of σ satisfies φ.
σ satisfies $(\exists v_i)\varphi$ iff some v_i-variant of σ satisfies φ.

A sentence is true in $\mathfrak{A}$ iff it's satisfied by every variable assignment, false in $\mathfrak{A}$ iff it's satisfied by none of them.
Every sentence is either true or false under $\mathfrak{A}$.

φ is a *theorem of logic* iff it's derivable from the axioms by the rules.

Define $\Gamma \vdash \varphi$ iff φ is a tautological consequence of $\Gamma \cup$ {theorems of logic}. φ is said to be a *logical consequence* of Γ iff it's true in every model of Γ.

Verify that each of the axioms is true in every model and that the rules preserve the property of being true in every model. Verify also that any tautological consequence of sentences that are true in a model is true in the model. This gives us this:

Soundness theorem. If $\Gamma \vdash \varphi$, then φ is a logical consequence of Γ.

Completeness theorem. If φ is a logical consequence of Γ, then $\Gamma \vdash \varphi$.

This was proven by Gödel in his doctoral dissertation, by a different method that the one we'll use here. The method we'll use here is due to Leon Henkin.

Suppose $\Gamma \nvdash \chi$. We begin by adding infinitely many constants to the language, listing the constants of the extended lanague as c_0, c_1, c_2,.... We list the sentences of the extended language as ξ_0, ξ_1, ξ_2,....

We extend Γ to a complete story Γ_∞ with $\Gamma_\infty \nvdash \chi$. Let $\Gamma_0 = \Gamma$. Given Γ_n with $\Gamma_n \nvdash \chi$, form Γ_{n+1} as follows:

If $\Gamma_n \cup \{\xi_n\} \vdash \chi$, let $\Gamma_{n+1} = \Gamma_n$.

If $\Gamma_n \cup \{\xi_n\} \nvdash \chi$ and ξ_n isn't existential, $\Gamma_{n+1} = \Gamma_n \cup \{\xi_n\}$.

If $\Gamma_n \cup \{\xi_n\} \nvdash \chi$ and ξ_n has the form $(\exists v_j)\psi(v_j)$, take the least i such that c_i doesn't appear in $\Gamma_n \cup \{\psi(v_j), \chi\}$, and let Γ_{n+1} be $\Gamma_n \cup \{\xi_n, \psi(c_i)\}$.

Γ_∞ = the union of the Γ_ns.

Define a model $\mathfrak{A}$ as follows:

$\mathfrak{A}(c_j)$ = the least i such that $c_i = c_j$ is in Γ_∞.
$|\mathfrak{A}|$ = the set of all the $\mathfrak{A}(c_j)$s.
$\langle \mathfrak{A}(c_{i_1}),..., \mathfrak{A}(c_{i_m})\rangle$ is in $\mathfrak{A}(R)$ iff $R(c_{i_1},..., c_{i_m})$ is in Γ_∞.

Show, by induction on complexity, that a sentence is in Γ_∞ iff it's true in $\mathfrak{A}$. Consequently, $\mathfrak{A}$ is a model in which all the members of Γ are true and χ is false.⊠

Corollary (Compactness theorem). If χ is a logical consequence of Γ, it's a logical consequence of some finite subset of Γ.

Corollary (Löwenheim-Skolem theorem). If Γ is consistent (that is, it has a model), it has a model whose domain is a set of natural numbers.