Constant-domain Modal Logics

A constant-domain model is an ordered sextuple <W,R,U,N,I,@>, where:

W is the set of worlds;
$R \subseteq W \times W$ is the accessibility relation;
U, which is nonempty, is the universe;
N, the naming function, takes constants to members of U;
I, the interpretation function, takes pairs <R,w>, where R is an n-place predicate and $w \in W$, to a set of n-tuples from w, subject to the condition that $\langle =,w\rangle = \{\langle x,x\rangle : x \in W\}$; and
$@ \in W$ is the actual world.

A variable assignment assigns an element of W to every world.
σ satisfies $R(\tau_1,\tau_2,...,\tau_n)$ in w iff $\langle b_1,b_2,..,b_n\rangle \in I(R,w)$, where $b_i = N(\tau_i)$ if τ_i is a constant, $\sigma(\tau_i)$ if τ_i isa variable.
σ satisfies a disjunction in a world iff it satisfies one or both disjuncts in that world, a conjunction iff it satisfies both conjuncts, and a negation iff it fails to satisfy the negatum.
A v_i-variant of σ is a variable assignment that agrees with σ except possibly in the value it assigns to v_i.
σ satisfies $(\forall v_i)\varphi$ in w iff every v_i-variant of σ satisfies φ w.
σ satisfies $(\exists v_i)\varphi$ in w iff some v_i-variant of σ satisfies φ in w.
σ satisfies $\Box\, \varphi$ in w iff σ satisfies φ in every world v with wRv.
σ satisfies $\Diamond\, \varphi$ in w iff σ satisfies φ in some world v with w R v.

A sentence is true in w iff it's satisfied by every variable assignment, false in w iff it's satisfied by none of them. A sentence is true in the model iff it's true in @.

Axioms for models with constant domain are sentences of the following forms:

(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
($\forall$Dist)	$(\forall v_i)(\varphi \rightarrow \psi) \rightarrow ((\forall v_i)\varphi \rightarrow (\forall v_i)\psi)$
(US)	$(\forall v_i)\varphi(v_i) \rightarrow \varphi(c)$
(Vac)	$(\varphi \leftrightarrow (\forall v_i)\varphi)$, v_i not free in φ
($\exists$Def)	$(\exists v_i)\varphi(v_i) \leftrightarrow \sim (\forall v_i) \sim \varphi(v_i)$
(Ref=)	$c = c$
(Sub=)	$(c=d \rightarrow (\varphi(c) \leftrightarrow \varphi(d))$
(BF)	$(\forall v_i)\Box\varphi \rightarrow \Box(\forall v_i)\varphi$
($\Box\neq$)	$\sim v_i = v_j \rightarrow \Box \sim v_i = v_j$

Rules:
TC
Nec

A sentence is a theorem of logic iff it's derivable from the axioms by the rules.We want to show that a sentence is a theorem of logic if and only if it is true in every model. First some preliminaries:

($\Box$=), which is ($\Box\neq$) with "$\neq$" replaced by "=," is derivable:

1	$v_i = v_i$	(Ref=)
2	$\Box\ v_i = v_i$	Nec 1
3	$v_i = v_j \to (\Box v_i = v_i \leftrightarrow \Box v_i = v_{j)}$	(Sub=)
4.	$v_i = v_j \to \Box v_i = v_j$	TC 2, 3

(CBC), the converse of (BC) is derivable:

1.	$(\forall x)\varphi(x) \to \varphi(c)$	(US)
2.	$\Box((\forall x)\varphi(x) \to \varphi(c)$	Nec 1
3.	$\Box(\forall x)\varphi(x) \to \Box\varphi(c)$	K 2
4.	$(\forall x)(\Box(\forall x)\varphi(x) \to \Box\varphi(x))$	UG 3
5.	$((\forall x)\Box(\forall x)\varphi(x) \to (\forall x)\Box\varphi(x))$	From 4 by ($\forall$Dist) and TC
6.	$\Box(\forall x)\varphi(x) \leftrightarrow (\forall x)\Box(\forall x)\varphi(x))$	(Vac)
7.	$\Box(\forall x)\varphi(x) \to (\forall x)\Box\varphi(x)$	TC 5, 6

Thc Barcan formula, (BF), isn't derivable from the other axioms, but it would be derivable if we expanded our axiom system to include (B): the proof is due to Arthur Prior:

1.	$(\forall x)\Box\varphi(x) \to \Box\varphi(c)$	US
2.	$\sim \Box\varphi(c) \to \sim (\forall x)\Box\varphi(x)$	TC 1
3.	$\Box(\sim \Box\varphi(c) \to \sim (\forall x)\Box\varphi(x))$	Nec 2
4.	$\Box\sim\Box\varphi(c) \to \Box\sim(\forall x)\Box\varphi(x))$	K3
5.	$\Diamond(\forall x)\Box\varphi(x) \to \Diamond\Box\varphi(c)$	TC 4, Def. of "$\Diamond$"
6.	$\Diamond\Box\varphi(c) \to \varphi(c)$	(B)
7.	$\Diamond(\forall x)\Box\varphi(x) \to \varphi(c)$	TC 5, 6
8.	$(\forall x)(\Diamond(\forall x)\Box\varphi(x) \to \varphi(x))$	UG 7
9.	$(\forall x)\Diamond(\forall x)\Box\varphi(x) \to (\forall x)\varphi(x)$	($\forall$Dist) 8, TC
10.	$\Diamond(\forall x)\Box\varphi(x) \leftrightarrow (\forall x)\Diamond(\forall x)\Box\varphi(x)$	(VQ)
11.	$\Diamond(\forall x)\Box\varphi(x) \to (\forall x)\varphi(x)$	TC 9, 10
12.	$\Box(\Diamond(\forall x)\Box\varphi(x) \to (\forall x)\varphi(x))$	Nec 11
13.	$\Box\Diamond(\forall x)\Box\varphi(x) \to \Box(\forall x)\varphi(x)$	K 12
14.	$(\forall x)\Box\varphi(x) \to \Box\ \Diamond(\forall x)\Box\varphi(x)$	(B)
15.	$(\forall x)\Box\varphi(x) \to \Box(\forall x)\varphi(x)$	TC 13, 14

Define $\Gamma \vdash \varphi$ iff φ is a tautological consequence of $\Gamma \cup$ {theorems of logic}; we say that φ is derivable from Γ.We want to show that $\Gamma \vdash \varphi$ iff φ is true in every model of Γ.

The proof of soundness consists, as usual, in verifying that the axioms are satisfied by every variable assignment in every world in every model and that this property is preserved by the rules. The only part of the proof that needs attention is (BF). Suppose $(\forall x)\Box\varphi(x)$ is satisfied by σ in w. Take any world v accessible from w. Let ρ be an "x"-variant of σ. ρ satisfies $\Box\varphi(x)$ in w. So ρ satisfies $\varphi(x)$ in v. Since ρ was an arbitrary "x"-variant of σ, σ satisfies $(\forall x)\varphi(x)$ in v. Since v was an arbitrary world accessible from w, σ satisfies $\Box(\forall x)\varphi(x)$ in w.

The proof of completeness uses a canonical frame construction like the one we used for normal systems for the modal sentential calculus. The worlds will be *complete stories with witnesses*, complete stories with the additional property that, whenever an existential sentence $(\exists x)\psi(x)$ is in the set, there is a constant c such that $\psi(c)$ is in the set.

Suppose $\Gamma \nvdash \chi$. Add infinitely many new constants language, and list the constants that result a $c_0, c_1, c_2,$.... List the sentences of the extended language as $\xi_0, \xi_1, \xi_2,$....

We do this by defining a sequence $u_0 \subseteq u_1 \subseteq u_2 \subseteq$... of sets of sentences.

$u_0 = \Gamma$.

Given u_n a finite extension of Γ from which χ isn't derivable, define u_{n+1}:

If $u_n \cup \{\xi_n\} \vdash \chi$, $u_{n+1} = u_n$.

If $u_n \cup \{\xi_n\} \nvdash \chi$ and ξ_n isn't existential, $u_{n+1} = u_n \cup \{\xi_n\}$.

If $u_n \cup \{\xi_n\} \nvdash \chi$ and ξ_n has the form $(\exists x)\psi(x)$, take the least i such that c_i doesn't appear in u_n, $\psi(x)$, or χ, and let $u_{n+1} = u_n \cup \{\xi_n, \psi(c_i)\}$.

Let @ be the union of all the u_ns. Then @ is a complete story with witnesses that includes Γ and the theorems of logic and excludes χ.

Let W, the set of worlds, be the set of all complete stories with witnesses that include all the theorems of logic and all the identity statements and negated identity statements in @.

Define wRv iff whenever $\Box\varphi$ is in w, φ is in v.

$N(c_i)$ = the least number j with $c_i = c_j$ an element of @.

U = the set of all the $N(c_i)$s.

$\langle N(c_{i_1}),...,N(c_{i_m})\rangle$ is in I(R,w) iff $Rc_{i_1}...c_{i_n}$ is in w.

Now that we have our model, we conplete the completeness proof by proving what's called the truth lemma: A sentence is true in a world iff it's an element of the world. The proof is by induction on the complexity of sentences. All the steps are routine, except for this:

$\Box\eta$ is in a world w iff η is true in every world accessible from w.

The left-to-right direction is immediate. We have to prove the right-to-left. That is, we assume that $\Box\eta$ isn't in the world w and show that there is a world v accessible from w in which η is false. By inductive hypothesis, it's enough to show that is a world accessible from w that doesn't contain η. That is, we want a complete story with witnesses that contains all the theorems of logic, the identity statements and negated identity sentences in @, and includes all the sentences θ with $\Box\theta$ in w, and excludes η. If θ is a $_{\text{the0orem}}$ of logic, $\Box\theta$ is a theorem of w, by Nec, so $\Box\theta$ is in w. If c=d is in @, c=d is in w, so $\Box$c=d is in w by ($\Box$=). Similarly, if ~ c=d is in @, $\Box$~ c=d is in w. So it will be enough to find a complete story with witnesses v that contains all the sentences θ with $\Box\theta$ in w and excludes η. We build v in stages.

v_0 = {sentences θ with $\Box\theta$ in w}.

If η were derivable from v_0, η would be a tautological $_{\text{consequence}}$ of {sentences θ with $\Box\theta$ in w}$\cup$ {theorems of logic}. If η were derivable from v_0, then by TC, Nec, and (K), $\Box\eta$ would be derivable from w, and so an element of w, contrary to hypothesis.

Given v_n a finite extension of v_0 from which η is not derivable, define v_{n+1} as follows:

If $v_n \cup \{\xi_n\} \vdash \eta$, $v_{n+1} = v_n$.

If $v_n \cup \{\xi_n\} \nvdash \eta$ and ξ_n isn't existential, $v_{n+1} = v_n \cup \{\xi_n\}$.

If $v_n \cup \{\xi_n\} \nvdash \eta$ and ξ_n has the form $(\exists x)\psi(x)$, take the least i such that $v_n \cup \{\psi(c_i)\} \nvdash \eta$ and let v_{n+1} be $v_n \cup \{\xi_n, \psi(c_i)\}$.

How do we know there is such a c_i? Suppose otherwise, and let δ be the conjunction of the sentences in $v_n \sim v_0$. (We've stipulated that the conjunction of the empty set is $\top$.) Then for each i, $(\delta \rightarrow (\psi(c_i) \rightarrow \eta))$ is derivable from $\{\theta: \Box\theta \in w\}$. So for each i, $(\delta \rightarrow (\sim \eta \rightarrow \sim \psi(c_i))$ is derivable from $\{\theta: \Box\theta \in w\}$. Consequently $\Box(\delta \rightarrow (\sim \eta \rightarrow \sim \psi(c_i)))$ is derivable from w, and so an element of w. So there is no constant c with $\sim \Box(\delta \rightarrow (\sim \eta \rightarrow \sim \psi(c)))$ an element of w. Since w is a complete story with witnesses $(\exists x)\sim\Box(\delta \rightarrow (\sim\eta \rightarrow \sim\psi(x)))$ isn't in w. So $(\forall x)\Box(\delta \rightarrow (\sim \eta \rightarrow \sim\psi(x)))$ is in w. By the Barcan formula, $\Box(\forall x)(\delta \rightarrow (\sim \eta \rightarrow \sim \psi(x)))$ is in w, and so $(\forall x)(\delta \rightarrow (\sim\eta \rightarrow \sim \psi(x)))$, which is logically equivalent to $(\delta \rightarrow (\sim \eta \rightarrow (\forall x)\sim \psi(x)))$ is in v_0, So $(\sim \eta \rightarrow (\forall x)\sim \psi(x)))$ is derivable from v_n and η is derivable from $v_n \cup \{(\exists x)\psi(x)\}$. Contradiction.

Finally, we take v to be the union of the v_ns. It will be a world accessible from w in which η is false.⊠

If we take our axioms and add any combination of schemata (T), (4), (B), and (5), we get a system that is sound and complete for models that satisfy the corresponding combination of being reflexive, transitive, symmetric, and transitive.