24.244. Modal Logic. Answers to the 8th p-set.

1. Show, by giving a derivation, that $((\varphi \vee \psi) > \sim \varphi) \vee (\varphi \vee \psi) > \sim \psi) \vee$ $(((\varphi \vee \psi) > \theta \leftrightarrow ((\varphi > \theta) \wedge (\psi > \theta)))$ is a theorem of Lewis's system.

1.	$((\varphi \vee \psi) > \sim \varphi) \vee (((\varphi \vee \psi) > \theta) \rightarrow (\varphi > \theta))$	Elimination of disjunctive antecedents
2.	$((\psi \vee \varphi) > \sim \psi) \vee (((\psi \vee \varphi) > \theta) \rightarrow (\psi > \theta))$	Elimination of disjunctive antecedents
3.	$(\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$	TC
4.	$(\varphi \vee \psi) > ((\varphi \vee \psi) \rightarrow (\psi \vee \theta))$	Conditionalization 3
5.	$(\varphi \vee \psi) > (\psi \vee \varphi)$	From 4 by conditional K, reflexive law, TC
6.	$(\psi \vee \varphi) > (\varphi \vee \psi)$	Substituting into 5
7.	$((\varphi \vee \psi) > \sim \psi) \leftrightarrow ((\psi \vee \varphi) > \sim \psi)$	From 6 by equivalent antecedents, TC
8.	$((\varphi \vee \psi) > \theta) \leftrightarrow ((\psi \vee \varphi) > \theta)$	From 6 by equivalent antecedenta, TC
9.	$((\varphi \vee \psi) > \sim \varphi) \vee ((\varphi \vee \psi) > \sim \psi) \vee (((\varphi \vee \psi) > \theta) \rightarrow ((\varphi > \theta) \wedge (\psi > \theta))$	TC1,2,7,8
10.	$((\varphi > \theta) \wedge (\psi > \theta)) \rightarrow ((\varphi \vee \psi) > \theta)$	Introduction of disjunctive antecedents
11.	$((\varphi \vee \psi) > \sim \varphi) \vee ((\varphi \vee \psi) > \sim \psi) \vee (((\varphi \vee \psi) > \theta) \leftrightarrow ((\varphi > \theta) \wedge (\psi > \theta))$	TC 9,10

2. Show that the instances of the schema $((\varphi > \theta) \wedge (\psi > \theta)) \rightarrow ((\varphi \vee \psi) > \theta)$ are true in every Lewis sphere model.

Suppose $(\varphi > \theta) \wedge (\psi > \theta)$ is true in @. There are four cases.

Case 1. There is no sphere in \$(@) in which either φ or ψ is true. Then there is no sphere in \$(@) in which $(\varphi \vee \psi)$ is true, so $((\varphi \vee \psi) > \theta$ is true in @..

Case 2. There is a sphere in \$(@) in which φ is true and no sphere in \$(@) in which ψ is true. Then there is a sphere S in \$(@) that contains a world in which φ is true and no worlds in which φ is true and θ is false. So there is a world in S in which $(\varphi \vee \psi)$ is true and and no world is S in which $(\varphi \vee \psi)$ is true and θ false. So $(\varphi \vee \psi) > \theta$ is true in @.

Case 3. There is no sphere in \$(@) in which φ is true but there are spheres in \$(@) in which ψ is true. Similar to case 2.

Case 4. There is a sphere in \$(@) in which there is a world in which φ is true and also a sphere in which there is a world in which ψ is true. Then there is a sphere S in \$(@) in which there is at least one φ-world w and in which there are no φ-worlds that aren't θ-worlds. Also that is a sphere T in \$(@) in which there is at least there is at least one ψ-world v and in which there are not ψ worlds that aren't θ-worlds. Because the spheres in \$(@) are nested, at least one of the following two subcases holds:

Subcase 4a. $S \subseteq T$. w is a world in S in which $(\varphi \vee \psi)$ holds. If u is a world in S in which $(\varphi \vee \psi)$ holds, then either φ is true in u, in which case θ is true in u, or ψ is true in u, in which, since u in in T, θ is true in u. Either way, θ is true in every $(\varphi \vee \psi)$-world in S, and so $(\varphi \vee \psi) > \theta$ is true in @.

Subcase 4b. $T \subseteq S$. Similar.

3. Is "$(((P \wedge Q) > R) \rightarrow ((P > R) \vee (Q > R)))$" a theorem of Stalnaker's system? Explain.

No. Take a model with four world @, w, v, and u, with w closer to @ than v and v closer to @ than u. "P" is true in w and u, "Q" is true in v and u, and "R" is true in u only. f(@,P) = w, f(@,Q) = v, and f(@,(P ∧ Q)) = u. "$((P \wedge Q) > R)$" is true in @ and "$(P > R)$" and "$Q > R)$" are both false in @.

4. **If "$(((P > R) \vee (Q > R)) \rightarrow ((P \wedge Q) > R))$" a theorem of Stalnaker's system? Explain.**
No. Use the same model as in problem 3, except that this time let "R" be true in every world but u. "$(P > R)$" and "$(Q > R)$" are both true in @ and "$((P \wedge Q) > R)$" is not.

5. **The law of exportation is the schema $(((\varphi \wedge \psi) > \theta) \rightarrow (\varphi > (\psi > \theta)))$. Show that, if we add it to Stalnaker's axioms, $((\varphi > \psi) \leftrightarrow (\varphi \rightarrow \psi))$ will become derivable.**

1.	$((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \psi$	TC
2.	$((\varphi \rightarrow \psi) \wedge \varphi) > (((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \psi)$	Conditionalization 1
3.	$(((\varphi \rightarrow \psi) \wedge \varphi) > ((\varphi \rightarrow \psi) \wedge \varphi)) \rightarrow (((\varphi \rightarrow \psi) \wedge \varphi) > \psi)$	From 2 by (Conditional K) and TC
4.	$((\varphi \rightarrow \psi) \wedge \varphi) > \psi$	TC 2, 3
5.	$(((\varphi \rightarrow \psi) \wedge \varphi) > \psi) \rightarrow ((\varphi \rightarrow \psi) > (\varphi > \psi))$	Exportation
6.	$((\varphi \rightarrow \psi) > (\varphi > \psi))$	TC 4,5
7.	$(((\varphi \rightarrow \psi) > (\varphi > \psi)) \wedge (\varphi \rightarrow \psi)) \rightarrow (\varphi > \psi)$	Modus ponens
8.	$(\varphi \rightarrow \psi) \rightarrow (\varphi > \psi)$	TC 6, 7
9.	$((\varphi > \psi) \wedge \varphi) \rightarrow \psi)$	Modus ponens
10.	$(\varphi > \psi) \rightarrow (\varphi \rightarrow \psi)$	TC 9
11.	$((\varphi > \psi) \leftrightarrow (\varphi \rightarrow \psi))$	TC6, 10

6. **The law of importation is the schema $((\varphi > (\psi > \theta)) \rightarrow ((\varphi \wedge \psi) > \theta))$. Show that, if we add it to Stalnaker's axioms, $((\varphi > \psi) \leftrightarrow (\varphi \rightarrow \psi))$ will not become derivable.**
Take a model with two worlds @ and w, each of which has access to itself and w, interpreted so that "P" is true in w only and "Q" isn't true in either world.

We want to show that each instance of the law of importation is true in @. Suppose $(\varphi > (\psi > \theta))$ is true in @. There are three cases:
Case 1. $(\varphi \wedge \psi)$ is true in @. $(\psi > \theta)$ is true in f(@,φ) = @. So θ is true in f(@,ψ) = @. So $((\varphi \wedge \psi) > \theta)$ is true in @.
Case 2. $(\varphi \wedge \psi)$ isn' true in @, but it is true in w. Two subcases:
Subcase 2a. φ is true in @. The ψ is false in @ and true in w. $(\psi > \theta)$ is true in f(@,φ) = @. So θ is true in f(@,ψ) = w = f(@,(φ ∧ ψ)), so $((\varphi \wedge \psi) > \theta)$ is true in @.
Subcase 2b. φ is false in b. The $(\psi > \theta)$ is true in f(@,φ) = w, so θ is true in f(w,ψ) = w = f(@,(φ ∧ ψ)) and $((\varphi \wedge \psi) > \theta)$ is true in @.
Case 3. $(\varphi \wedge \psi)$ isn't true in either world. The $((\varphi \wedge \psi) > \theta)$ is true in @.

An exactly symmetrical argument shows that each instance of the law of importation is true in w.

Every instance of importation is true in both worlds of our model. Each of the axioms of logic is true in every world of every model, hence true in both worlds of the model we are considering here. Moreover, the class of sentences true in both worlds in our model is closed under TC and Conditionalization. It follows that every sentence derivable by the rules from the extended system of axioms is true at both worlds in our model. However, "$((P > Q) \leftrightarrow (P \rightarrow Q))$" isn't true at @ in our model, so it isn't among the sentences derivable by the rules form the extended axiom system

7. Let Γ be smallest collection of sentences of the language obtained from the language of arithmetic by adding the new predicate "Nec" that:

> **contains all the consequences of Q and all sentences of the form $\mathrm{Nec}([\ulcorner \theta \urcorner])$ for θ a consequence of Q;**
> **contains all sentences of the form $(\mathrm{Nec}([\ulcorner (\varphi \rightarrow \psi) \urcorner] \rightarrow (\mathrm{Nec}([\ulcorner \varphi \urcorner] \rightarrow \mathrm{Nec}([\ulcorner \psi \urcorner]))$; and**
> **contains all sentences of the forms $(\mathrm{Nec}([\ulcorner \varphi \urcorner]) \rightarrow \varphi)$ and $\mathrm{Nec}([\ulcorner (\mathrm{Nec}([\ulcorner \varphi \urcorner]) \rightarrow \varphi) \urcorner])$.**

Show that the formalization of Goldbach's conjecture ("Every even number > 2 is the sum of two primes") is a consequence of Γ.

We can use the Self-reference lemma to find a sentence ν such that $(\nu \leftrightarrow \sim \mathrm{Nec}([\ulcorner \nu \urcorner]))$ is a consequence of Q. The following sentences are consequences if Γ:

1.	$\mathrm{Nec}([\ulcorner (\sim \mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \urcorner])$	Result of prefixing "Nec" to a consequence of Q.
2.	$\mathrm{Nec}([\ulcorner (\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \urcorner])$	Prefixing "Nec" to an instance of (T)
3.	$\mathrm{Nec}([\ulcorner (\sim \mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \rightarrow ((\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \rightarrow \nu)) \urcorner])$	Prefixing "Nec" to a theorem of Q
4.	$(\mathrm{Nec}([\ulcorner (\sim \mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \rightarrow ((\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \rightarrow \nu)) \urcorner]) \rightarrow (\mathrm{Nec}([\ulcorner (\sim \mathrm{Nec}([\ulcorner \nu \urcorner] \rightarrow \nu) \urcorner]) \rightarrow \mathrm{Nec}([(\ulcorner (\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \rightarrow \nu)) \urcorner])))$	(K)
5.	$(\mathrm{Nec}([\ulcorner (\sim \mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \urcorner]) \rightarrow \mathrm{Nec}([(\ulcorner (\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \rightarrow \nu) \urcorner]))$	TC 3, 4
6.	$\mathrm{Nec}([(\ulcorner (\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \rightarrow \nu) \urcorner])$	TC 1, 5
7.	$(\mathrm{Nec}([(\ulcorner (\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \rightarrow \nu) \urcorner]) \rightarrow (\mathrm{Nec}([\ulcorner (\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu) \urcorner]) \rightarrow \mathrm{Nec}([\ulcorner \nu \urcorner])))$	(K)
8.	$(\mathrm{Nec}([\ulcorner (\mathrm{Nec}([\ulcorner \nu \urcorner] \rightarrow \nu) \urcorner]) \rightarrow \mathrm{Nec}([\ulcorner \nu \urcorner]))$	TC 6, 7
9.	$\mathrm{Nec}([\ulcorner \nu \urcorner])$	TC 2, 8
10.	$(\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \nu)$	Instance of (T)
11.	ν	TC 9, 10
12.	$(\mathrm{Nec}([\ulcorner \nu \urcorner]) \rightarrow \sim \nu)$	Consequence of Q
13.	$\sim \nu$	TC 9, 12
14.	Goldbach's conjecture	TC 11, 13