## Variable-domain Modal Logics

A modal predicate logic that requires that the same individuals exist in every world is not very satisfactory. The Barcan formula assures us that, if there could be things of a certain kind, then there are actual things that could be of that kind. That doesn't seem right. There could be unicorns, but there aren't any actual things that could be unicorns. The converse Barcan property tells us that is there exists a thing that might have a certain property, then it's possible that there's something that has the property. That seems reasonable, but it has an unwelcome consequence. "$\Box(\forall x)(\exists y)x = y$" is obtained by necessitation from a theorem of the predicate calculus. By the converse Barcan formula, this leads to "$(\forall x)\Box(\exists y)x=y$," which says that each actual individual exists necessarily.

The Barcan formula is an axiom. If we don't like it, we can remove it from our list of axioms. Its converse is harder to get rid of. The way we get an axiom system for modal predicate calculus is to take our system of axioms and rules for modal sentential calculus and our system of axioms and rules for the plain predicate calculus and combine them. For the version of the plain predicate calculus without individual constants, this is a typical set of axioms:

(Taut) Every tautological formula.

(US) $(\forall v_i)\varphi(v_i) \rightarrow \varphi(v_j)$, where $\varphi(v_j)$ is like $\varphi(v_i)$, except for containing free $v_j$ at some places where $\varphi(v_i)$ contains free $v_i$.

($\forall$Dist) $(\forall v_i)(\varphi \rightarrow \psi) \rightarrow ((\forall v_i)\varphi \rightarrow (\forall v_i)\psi)$.

(Vac) $\varphi \leftrightarrow (\forall v_i)\varphi$, provided $v_i$ isn't free in $\varphi$.

($\exists$Def) $(\exists v_i)\varphi \leftrightarrow \sim (\forall v_i) \sim \varphi$.

(Ref=) $v_i = v_j$.

(Sub=) $v_i = v_j \rightarrow (\varphi(v_i) \leftrightarrow \varphi(v_j)$, where $\varphi(v_j)$ is like $\varphi(v_i)$ except for containing free $v_j$ at some places where $\varphi(v_i)$ has free $v_i$.

The rules will include:

UG From $\varphi$, you may infer $(\forall v_i)\varphi$,

as well as modus ponens.The modal system will include (K) at minimum, and may optionally contain such other axioms as (T) and (4). It will contain Nec as a rule, and hence K and TC as derived rules

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This combination lets us derive the converse Barcan formula (using "x" instead of "$x_i$" to avoid subscripts):

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|---|---|---|
| 1. | $(\forall x)\varphi(x) \rightarrow \varphi(x)$ | (US) |
| 2. | $\Box((\forall x)\varphi(x) \rightarrow \varphi(x))$ | Nec 1 |
| 3. | $\Box(\forall x)\varphi(x) \rightarrow \Box\varphi(x)$ | K 2 |
| 4. | $(\forall x)(\Box(\forall x)\varphi(x) \rightarrow \Box\varphi(x))$ | UG 3 |
| 5. | $(\forall x)\Box(\forall x)\varphi(x) \rightarrow (\forall x)\Box\varphi(x)$ | From 4 by ($\forall$Dist) |
| 6. | $\Box(\forall x)\varphi(x) \leftrightarrow (\forall x)\Box(\forall x)\varphi(x)$ | (Vac) |
| 7. | $\Box(\forall x)\varphi(x) \rightarrow (\forall x)\Box\varphi(x)$ | TC 5, 6 |

This result looks inevitable, but is it isn't really. There are many different axiomatizations of the predicate calculus, which get to the same theorems by different paths. Thus if we replace (US) by the following , we get an equivalent axiomatization of the predicate calculus:

$(\forall\text{US})\ (\forall v_j)((\forall v_i)\varphi(v_i) \rightarrow \varphi(v_j))$.
As long as we're just doing predicate calculus, one system is exactly as good as the other, but when we combine the two systems with the modal axioms and rules, there are significant differences. In the new system, the converse Barcan formula is no longer derivable. The same maneuver thwarts the derivation in KB of the Barcan formula.

That the converse Barcan formula is avoidable is a key result of Kripke's 1963 paper, "Semantical Considerations on Modal Logic." The system of axioms for the predicate calculus he talked about there, which was developed by Quine in *Mathematical Logic*, didn't contain UG. Instead, it replaced all the axioms with their universal closures and used modus ponens as its only rule. In combining predicate calculus with modal logic, the axioms will be obtained from he schemata by prefixing arbitrary strings of universal quantifiers and "□"s. In the system we develop below, we'll achieve the same effect by adopting a more complicated form of UG.

Quine's system from *Mathematical Logic* didn't contain individual constants. But it's useful to have constants, so the predicate-calculus axioms we'll build on here will be the axioms for free logic

A model will be an ordered septuple $<W,R,U,D,N,I,@>$ , where $<W,R,U,N,I,@>$ is a constant-domain model and D is a function that assigns to each world a subset of U as the domain of that world.

The only change required in the definition of satisfaction is the clauses for the quantifiers:

> $\sigma$ satisfies $(\exists v_i)\varphi$ in w iff $\varphi$ is satisfied by a variable assignmen that agrees with $\sigma$ except possibly in the value it assigns to $v_i$ and that assigns a member of D(w) to $v_i$.
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> $\sigma$ satisfies $(\forall v_i)\varphi$ in w iff $\varphi$ is satisfied by every variable assignment that agrees with $\sigma$ except possibly in the value it assigns to $v_i$ and that assigns a member of D(w) to $v_i$.

The basic axiom system is obtained rom the axioms for free logic by adding the modal axioms (K) and $(\Box\neq)$. Other modal axioms – (T), (4), (5), and so on – can be added at will. The rules are TC, UG, Nec, and a new rule:

$\text{UG}^n$ From $(\psi_1 \rightarrow \Box(\psi_2 \rightarrow \Box(\psi_3 \rightarrow\ldots \Box(\psi_{n-1} \rightarrow \Box\psi_n(c))\ldots)))$, you may infer $(\psi_1 \rightarrow \Box(\psi_2 \rightarrow \Box(\psi_3 \rightarrow\ldots\ \Box(\psi_{n-1} \rightarrow \Box(\forall x)\varphi_n(x))\ldots)))$, provided c doesn't appear within any of the $\psi_i$s for $1 \leq i < n$ or in $\psi_n(x)$ and x occurs free in $\psi_n(x)$.

In showing that $\text{UG}^n$ preserves validity, we appeal to the following:

> **Lemma.** The conditional $(\psi_1 \rightarrow \Box(\psi_2 \rightarrow \Box(\psi_3 \rightarrow\ldots \rightarrow \Box(\psi_{n-1} \rightarrow \Box\psi_n)\ldots)))$ is true in the model $<W,R,U,D,N,I,w_1>$ iff, for each sequence $w_1, w_2, w_3,\ldots, w_{n-1}, w_n$, with each $w_{i+1}$ accessible from $w_i$ and with $\psi_i$ true in $w_i$ for $1 \leq i < n$, $\psi_n$ is true in $w_n$.

The proof is by an easy induction on n.

Suppose $(\psi_1 \rightarrow \Box(\psi_2 \rightarrow \Box(\psi_3 \rightarrow \ldots \Box(\psi_{n-1} \rightarrow \Box\psi_n(c))\ldots)))$ is valid. Given a model $<W,R,U,D,N,I,w_1>$ and a sequence $w_1, w_2, w_3, \ldots, w_n, w_{n+1}$, with each $w_{i+1}$ accessible from $w_i$ and with $\psi_i$ true in $w_i$ for $1 \leq i < n$, we want to see that $(\forall x)\psi_n(x)$ is true in $w_{n+1}$, that is, that it is satisfied in $w_{n+1}$ by every variable assignment. Take a variable assignment $\sigma$. Let $\rho$ be a variable assignment that agrees with $\sigma$ except possibly in the value it assigns to "x" and that assigns to "x" an element of $D(w_n)$. We want to see that $\rho$ satisfies $\psi_n(x)$ in $<W,R,U,D,N,I,w_n>$. Let N* be just like N except that $N^*(c) = \rho(x)$. Since $(\psi_1 \rightarrow \Box(\psi_2 \rightarrow \Box(\psi_3 \rightarrow \ldots \Box(\psi_{n-1} \rightarrow \Box\psi_n(c))\ldots)))$ is true in $<W,R,U,D,N^*,I,w_1>$, $\psi_n(c)$ is true in $<W,R,U,D,N^*,I,w_n>$. So $\rho$ satisfies $\psi_n(x)$ in $<W,R,U,D,N^*,I,w_n>$, and hence in $<W,R,U,D,N,I,w_n>$.

$UG^n$ isn't a principle that leaps to mind when you think about modal reasoning. The discovery that it's the key to getting canonical models for modal predicate calculus with variable domains is due to Richmond Thomason.

We'll need a specialized version of the familiar notion of complete story. A *complete modal story* is a complete story that:

> contains the sentences derivable from the axioms by the rules;
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> contains $(\exists y)y = c$ and $\varphi(c)$, for some c, whenever it contains $(\exists x)\varphi(x)$; and
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> contains $\Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge \ldots \Diamond(\psi_{n-1} \wedge \Diamond((\exists y)y = c \wedge \psi_n(c)))\ldots)))$, for some c, whenever it contains contains $\Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge \ldots \Diamond(\psi_{n-1} \wedge \Diamond(\exists x)\psi_n(x))\ldots)))$, where c doesn't appear in $\psi_n(x)$ or any of the $\psi_i$s for $1 \leq i < n$ and "x" is free in $\psi_n(x)$.
>
> **Lemma.** If $\Gamma \nvdash \chi$, there is, within the language obtained from the language of $\Gamma$ and $\chi$ by adding infinitely many constatnts, a modal complete story that contains $\Gamma$ and excludes $\chi$.

**Proof :** Let $\Gamma_0 = \Gamma$. Given $\Gamma_n$, we form $\Gamma_{n+1}$ as follows, listing the sentences as $\xi_0, \xi_1, \xi_2, \ldots$:

Case 1. If $\Gamma_n \cup \{\xi_n\} \vdash \chi$, $\Gamma_{n+1} = \Gamma_n$.

Case 2. If $\Gamma_n \cup \{\xi_n\} \nvdash \chi$ and $\xi_n$ has the form $(\exists x)\varphi(x)$, take the first constant c that doesn't appear in $\Gamma$, $\varphi(x)$, or $\chi$, and let $\Gamma_{n+1} = \Gamma_n \cup \{\xi_n, (\exists y)y=c, \varphi(c)\}$.

Case 3. If $\Gamma_n \cup \{\xi_n\} \nvdash \chi$ and $\xi_n$ has the form $\Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge \ldots \Diamond(\psi_{m-1} \wedge \Diamond(\exists x)\psi_m(x))\ldots)))$, take the first constant c that doesn't appear in $\Gamma$, $\xi_n$, or $\chi$, and let $\Gamma_{n+1}$ be $\Gamma_n \cup \{\xi_n, \Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge \ldots \Diamond(\psi_{m-1} \wedge \Diamond((\exists y)y=c \wedge \psi_m(c)))\ldots)))$.

Case 4. Otherwise, $\Gamma_{n+1} = \Gamma_n \cup \{\xi_n\}$.

How do we know that, in case 2, $\Gamma_{n+1} \nvdash \chi$? Otherwise, there would be $\theta_1,\ldots, \theta_k$ in $\Gamma_n$ so that $((\theta_1 \wedge \ldots \wedge \theta_k \wedge \xi_n \wedge \sim \chi) \to ((\exists y)y=c \to \sim \varphi(c)))$ is a theorem of logic. By UG, ($\forall$Dist), and (VQ), $((\theta_1 \wedge \ldots \wedge \theta_k \wedge \xi_n \wedge \sim \chi) \to (\forall x)((\exists y)y=x \to\sim \varphi(x)))$ is likewise a theorem. By (EE), $((\forall x)((\exists y)y=x \to \sim \varphi(x)) \to (\forall x){\sim}\varphi(x))$ is a theorem. So $((\theta_1 \wedge\ldots \wedge \theta_k \wedge \xi_n) \to \chi)$ is a theorem. Contradiction.

How do we know that, in case 3, $\Gamma_{n+1} \nvdash \chi$? Otherwise, there would be $\theta_1,\ldots, \theta_k$ in $\Gamma_n$ so that $(\theta_1 \wedge \ldots \wedge \theta_k \wedge \xi_n \wedge \sim \chi) \to \sim \Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge\ldots \Diamond(\psi_{m-1} \wedge \Diamond((\exists y)y=c \wedge \psi_m(c)))\ldots)))$ is a theorem of logic. So $(\theta_1 \wedge \ldots \wedge \theta_k \wedge \xi_n \wedge \sim \chi) \to \Box(\psi_1 \to \Box(\psi_2 \to \Box(\psi_3 \to\ldots \Box(\psi_{m-1} \wedge \to \Box((\exists y)y=c \to \sim \psi_m(c)))\ldots)))$ is a theorem. By $UG^{m+1}$, $(\theta_1 \wedge \ldots \wedge \theta_k \wedge \xi_n \wedge \sim \chi) \to \Box(\psi_1 \to \Box(\psi_2 \to \Box(\psi_3 \to\ldots \Box(\psi_{m-1} \wedge \to \Box(\forall x)((\exists y)y=x \to \sim \psi_m(x)))\ldots)))$ is a theorem. Since $((\forall x)((\exists y)y=x \to \sim \varphi(x)) \leftrightarrow \sim (\exists x)\varphi(x))$ is a theorem, we conclude that $(\theta_1 \wedge\ldots \wedge \theta_k) \to ((\xi_n \wedge \sim \chi) \to \sim \xi_n)$ is a theorem. Contradiction.

Our complete modal story will be the union of the $\Gamma_n$s.⊠

Given $\Gamma$ and $\chi$ with $\Gamma \nvdash \chi$, let @ be a complete modal story that includes $\Gamma$ and excludes $\chi$. Let W be the set of complete modal stories that include all the identity statements and negated identity statements in @. Define w R v iff, whenver $\Box\varphi$ is in w, $\varphi$ is in v. Listing the constants as $c_0$, $c_1$, $c_2$,..., Let $N(c_i)$ be the least j with $c_i = c_j$ in @. Let U be the range of N. $N(c_i)$ is in D(w) iff $(\exists y)y = c_i$ is in w. $\langle N(c_{i_1}),\ldots, N(c_{i_k})\rangle$ is in I(R,w) iff $Rc_{i_1}\ldots c_{i_k}$ is in w.

To complete the proof of the completeness theorem, we need to prove the Truth Lemma: A sentence is true in a world iff it's an element of the world. The proof is by induction on the complexity of sentences. The only part of the proof that isn't routine is this: If $\Box\eta$ isn't in w, then there is a world v accessible from w with $\eta \notin v$.

Let $v_0$ be the set of sentences $\theta$ with $\Box\theta \in w$. Then $v_0$ includes all the theorems of logic and all the identity statements and negated identity statements in @. $\eta$ isn't derivable from it. We want to build up a complete modal story including $v_0$ and excluding $\eta$. Assume we already have $v_n$ with $v_n \nvdash \eta$. There are four cases.

Case 1. $v_n \cup \{\xi_n\} \vdash \eta$. Then $v_{n+1} = v_n$.

Case 2. $v_n \cup \{\xi_n\} \nvdash \eta$ and $\xi_n$ has the form $(\exists x)\varphi(x)$. Find the first c such that $v_n \cup \{(\exists y)y=c, \varphi(c)\} \nvdash \eta$, and let $v_{n+1} = v_n \cup \{\xi_n, (\exists y)y=c, \varphi(c)$.

How do we know there is such a c? If not, then, where $\gamma$ = the conjunction of the members of $v_n \sim v_0$, there is, for each constant c, a sentence $\theta$ with $\Box\theta$ in w so that $(\theta \to \sim ((\exists y)y=c \wedge \gamma \wedge \sim \eta \wedge \varphi(c))$ is a theorem. So the result of putting a "$\Box$" in front of it is a theorem. So $\sim \Diamond ((\exists y)y =c \wedge (\gamma \wedge \sim \eta \wedge \varphi(c))$ is in w. So $\Diamond((\exists y)y = c \wedge (\gamma \wedge \sim \eta \wedge \varphi(c)))$ isn't in w, for any c. So, by the third condition in the definition of "complete modal story," with n=1, $\Diamond(\exists x)(\gamma \wedge \sim \eta \wedge \varphi(x))$ isn't in w. So $\Box \sim (\exists x)(\gamma \wedge \sim \eta \wedge\varphi(x))$ is in w. So $\sim (\exists x)(\gamma \wedge \sim \eta \wedge \varphi(x))$ is in $v_0$. So $(\exists x)\varphi(x) \to \eta)$ is derivable from $v_n$. Contradiction.

Case 3. $v_n \cup \{\xi_n\} \nvdash \eta$ and $\xi_n$ has the form $\Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge\ldots \Diamond(\psi_{m-1} \wedge \Diamond(\exists x)\psi_m(x))\ldots)))$. Take the first c so that $v_n \cup \{\Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge\ldots \Diamond(\psi_{m-1} \wedge \Diamond((\exists y)y = c \wedge \psi_m(c)))\ldots)))\} \nvdash \eta$, and let $v_{n+1} = v_n \cup \{\xi_n, \Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge ... \Diamond(\psi_{m-1} \wedge \Diamond((\exists y)y = c \wedge \psi_m(c)))...)))\}$.

How do we know there is such a c? If not, then, where $\gamma$ = the conjunction of the members of $v_n \sim v_0$, there is, for each constant c, a sentence $\theta$ with $\Box\theta$ in w so that $((\theta \wedge \gamma \wedge \sim \eta) \rightarrow \Box(\psi_1 \rightarrow \Box(\psi_2 \rightarrow \Box(\psi_3 \rightarrow ... \Box(\psi_{m-1} \rightarrow \Box((\exists y)y{=}c \rightarrow \sim \psi_m(c))...))))$ is a theorem. The result of putting a "$\Box$" in front of it is a theorem. So $\sim \Diamond((\theta \wedge \gamma \wedge \sim \eta) \wedge \Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge ... \Diamond(\psi_{m-1} \wedge \Diamond((\exists y)y{=}c \wedge \psi_m(c)))...))))$ is in w, for every c. Using the third clause in the definition of "complete modal story," $\sim \Diamond((\theta \wedge \gamma \wedge \sim \eta) \wedge \Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge ... \Diamond(\psi_{m-1} \wedge \Diamond(\exists x)\psi_m(x))...))))$ is in w. $\Box((\theta \wedge \gamma \wedge \sim \eta) \rightarrow \Box(\psi_1 \rightarrow \Box(\psi_2 \rightarrow \Box(\psi_3 \rightarrow ... \Box(\psi_{m-1} \rightarrow \Box{\sim}(\exists x)\psi_{m-1}(x))...))))$ is in w. The result of deleting the initial "$\Box$" is in $v_0$. $v_n \vdash (\Diamond(\psi_1 \wedge \Diamond(\psi_2 \wedge \Diamond(\psi_3 \wedge ... \Diamond(\psi_{m-1} \wedge \Diamond(\exists x)\psi_m(x))...))) \rightarrow \eta)$. Contradiction.

Case 4. Otherwise. Let $v_{n+1} = v_n \cup \{\xi_n\}$.

Let v be the union of the $v_n$s. Then v is a world accessible from w that doesn't contain $\eta$. By inductive hypothesis, $\eta$ is false in v.⊠

As usual, if we add any combination of the schemata (T), (4), (B), and (5) to the axioms, we'll get a sound and complete axiom system for the class of models whose accessibility relations are the corresponding combination of reflexive, transitive, symmetric, and Euclidean.