The Higher Infinite

Summary Sheet – 24.118, Spring 2021

1 The Ordinals

1.1 How We'll Get to the Ordinals

Ordering $\rightarrow$ Total Ordering $\rightarrow$ Well-Ordering $\rightarrow$ Well-Order Type $\rightarrow$ Ordinal

1.2 Orderings

Think of $x < y$ as meaning "x precedes y". We say that $<$ is an **ordering** on set A if and only if for any $a, b, c \in A$:

Asymmetry If $a < b$, then not-$(b < a)$.

Transitivity If $a < b$ and $b < c$, then $a < c$.

1.3 Total Orderings

A **total ordering** $<$ on A is an ordering on A such that for any distinct elements a, b of A:

Totality $a < b$ or $b < a$

1.4 Well-Orderings

A **well-ordering** $<$ of A is a total ordering on A such that:

Well-Ordering Every non-empty subset S of A has a $<$-smallest member.

1.5 Well-order types

The orderings $<_1$ and $<_2$ are of the same type if they are isomorphic.*

*Let $<_1$ be an ordering on A and $<_2$ be an ordering on B. Then $<_1$ is **isomorphic** to $<_2$ if and only if there is a bijection f from A to B such that, for every x and y in A, $x <_1 y$ if and only if $f(x) <_2 f(y)$.

1.6 The First Few Ordinals

ordinal	name of ordinal	well-order type represented
$\{\}$	0	
$\{0\}$	$0'$	$\vert$
$\{0, 0'\}$	$0''$	$\vert\vert$
$\{0, 0', 0''\}$	$0'''$	$\vert\vert\vert$
$\vdots$	$\vdots$	
$\{0, 0', 0'', 0''', \ldots\}$	ω	$\vert\vert\vert\ldots$
$\{0, 0', 0'', 0''', \ldots, \omega\}$	ω'	$\vert\vert\vert\ldots\vert$
$\{0, 0', 0'', 0''', \ldots, \omega, \omega'\}$	ω''	$\vert\vert\vert\ldots\vert\vert$
$\vdots$	$\vdots$	$\vdots$

1.7 Constructing the Ordinals

Construction Principle At each stage, we introduce a new ordinal, namely: the set of all ordinals that have been introduced at previous stages.

Open-Endedness Principle However many stages have occurred, there is always a "next" stage, that is, a first stage after every stage considered so far.[†]

1.8 Ordering the Ordinals

The ordinals are well-ordered by the following precedence relation:

$$\alpha <_o \beta \leftrightarrow_{df} \alpha \in \beta$$

1.9 Representing Well-Order Types

Since every ordinal is a set of ordinals, the elements of an ordinal are always well-ordered by $<_o$. So we may set forth the following:

Representation Principle Each ordinal represents the well-order type that it itself instantiates under $<_o$.

1.10 Some Definitions

- $\alpha' = \alpha \cup \{\alpha\}$
- A **successor ordinal** is an ordinal α such that $\alpha = \beta'$ for some β.
- A **limit ordinal** is an ordinal that is not a successor ordinal.

[†]It is important to interpret the Open-Endedness Principle as entailing that there is no such thing as "all" stages—and therefore deliver the result that there is no such thing as "all" ordinals.

2 Ordinal Addition

The intuitive idea: A well-ordering of type $(\alpha + \beta)$ is the result of starting with a well-ordering of type α and appending a well-ordering of type β at the end.

Formally:

$$\begin{array}{lllll} \alpha & + & 0 & = & \alpha \\ \alpha & + & \beta' & = & (\alpha + \beta)' \\ \alpha & + & \lambda & = & \bigcup\{\alpha + \beta : \beta < \lambda\} \ (\lambda \text{ a limit ordinal}) \end{array}$$

- Ordinal addition is associative: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
- Ordinal addition is *not* commutative: it is not generally the case that $\alpha + \beta = \beta + \alpha$.

3 Ordinal Multiplication

The intuitive idea: A well-ordering of type $(\alpha \times \beta)$ is the result of starting with a well-ordering of type β and replacing each position in the ordering with a well-ordering of type α.

Formally:

$$\begin{array}{lllll} \alpha & \times & 0 & = & 0 \\ \alpha & \times & \beta' & = & (\alpha \times \beta) + \alpha \\ \alpha & \times & \lambda & = & \bigcup\{\alpha \times \beta : \beta < \lambda\} \ (\lambda \text{ a limit ordinal}) \end{array}$$

- Ordinal multiplication is associative: $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$.
- Ordinal multiplication is *not* commutative: it is not generally the case that $\alpha \times \beta = \beta \times \alpha$.

4 Some Additional Operations

- Exponentiation:

$$\begin{array}{lll} \alpha^0 & = & 0' \\ \alpha^{\beta'} & = & (\alpha^\beta) \times \alpha \\ \alpha^\lambda & = & \bigcup\{\alpha^\beta : \beta < \lambda\} \ (\lambda \text{ a limit ordinal}) \end{array}$$

- Tetration:

$$\begin{array}{lll} {}^0\alpha & = & 0' \\ {}^{\beta'}\alpha & = & ({}^\beta\alpha)^\alpha \\ {}^\lambda\alpha & = & \bigcup\{{}^\beta\alpha : \beta < \lambda\} \ (\lambda \text{ a limit ordinal}) \end{array}$$

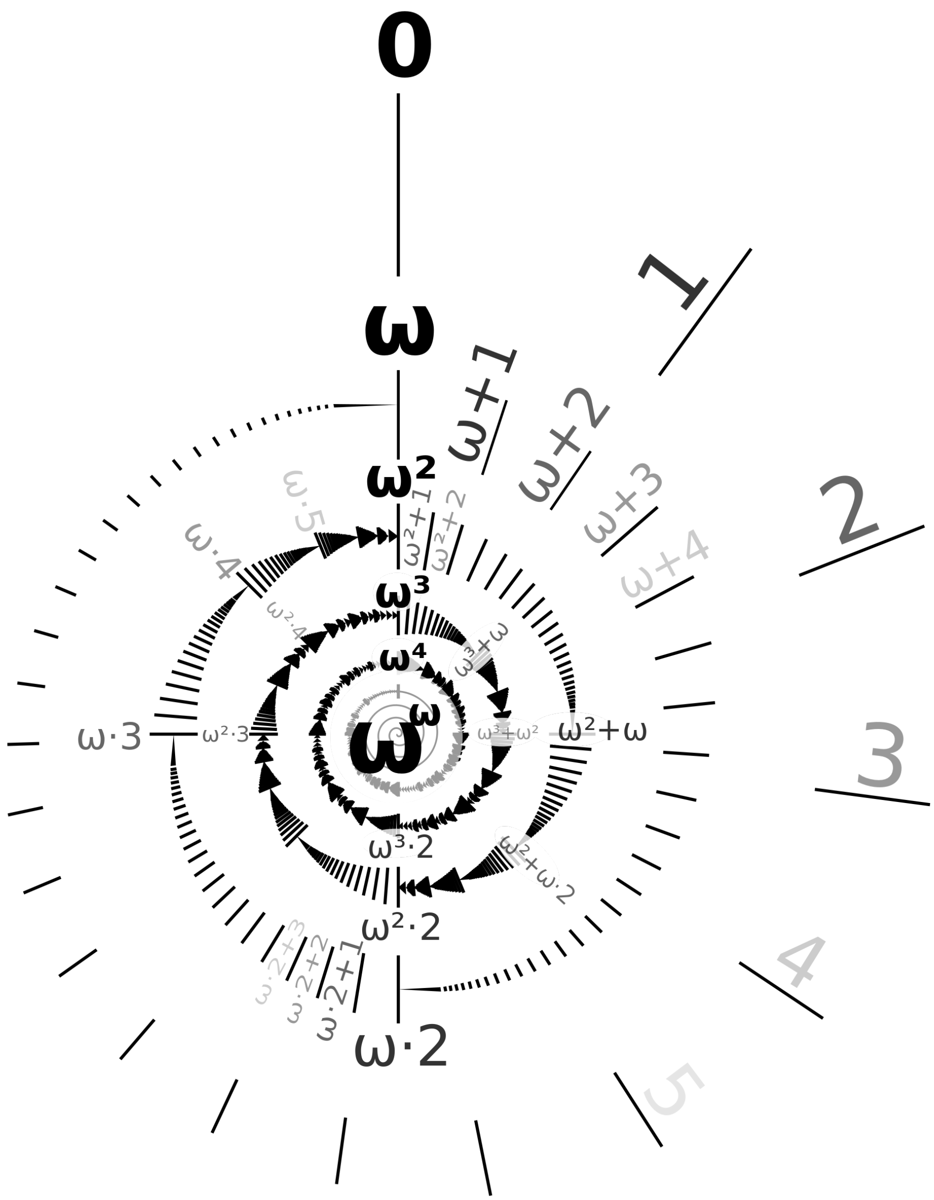
- And so forth...

5 Some Additional Ordinals

ordinal	members	well-order type represented
ω	$\{0, 0', \ldots\}$	$\|\|\|\ldots$
$\omega + 0'$	$\{0, 0', \ldots, \omega\}$	$\|\|\|\ldots\|$
$\omega + \omega$	$\{0, 0', \ldots, \omega, \omega + 0', \ldots\}$	$\|\|\|\ldots\|\|\|\ldots$
$\omega + \omega + \omega$	$\{0, 0', \ldots, \omega, \omega + 0', \ldots, \omega + \omega, \omega + \omega + 0', \ldots\}$	$\|\|\|\ldots\|\|\|\ldots\|\|\|\ldots$
$\omega \times \omega$ $= \omega^{0''}$	$\{0, \ldots, \omega, \ldots, \omega + \omega, \ldots, \omega + \omega + \omega, \ldots\ \ldots\}$	$\underbrace{\|\|\|\ldots\|\|\|\ldots\|\|\|\ldots\ \ldots}_{\omega\text{ times}}$
$(\omega \times \omega) + \omega$	$\{0, \ldots, \omega \times \omega, (\omega \times \omega) + 0', (\omega \times \omega) + 0'', \ldots\}$	$\underbrace{\|\|\|\ldots\|\|\|\ldots\|\|\|\ldots\ \ldots}_{\omega\text{ times}}\|\|\|\ldots$
$(\omega \times \omega) + \omega + \omega$	$\{0, \ldots, \omega \times \omega, (\omega \times \omega) + 0', \ldots (\omega \times \omega) + \omega, (\omega \times \omega) + \omega + 0' \ldots\}$	$\underbrace{\|\|\|\ldots\|\|\|\ldots\|\|\|\ldots\ \ldots}_{\omega\text{ times}}\|\|\|\ldots\|\|\|\ldots$
$(\omega \times \omega) + (\omega \times \omega)$ $= \ \omega \times \omega \times 0''$	$\{0, \ldots, \omega \times \omega, \ldots (\omega \times \omega) + \omega, \ldots, (\omega \times \omega) + \omega + \omega \ldots\ \ldots\}$	$\underbrace{\|\|\|\ldots\|\|\|\ldots\|\|\|\ldots\ \ldots}_{\omega\text{ times}}\ \underbrace{\|\|\|\ldots\|\|\|\ldots\|\|\|\ldots\ \ldots}_{\omega\text{ times}}$
$\omega \times \omega \times 0'''$	$\{0, \ldots, \omega \times \omega, \ldots (\omega \times \omega) + \omega, \ldots, (\omega \times \omega) + \omega + \omega \ldots\ \ldots$ $(\omega \times \omega) + (\omega \times \omega) \ldots (\omega \times \omega) + (\omega \times \omega) + \omega \ldots (\omega \times \omega) + (\omega \times \omega) + \omega + \omega \ldots\ \ldots\}$	$\underbrace{\|\|\|\ldots\|\|\|\ldots\ \ldots}_{\omega\text{ times}}\ \underbrace{\|\|\|\ldots\|\|\|\ldots\ \ldots}_{\omega\text{ times}}\ \underbrace{\|\|\|\ldots\|\|\|\ldots\ \ldots}_{\omega\text{ times}}$
$\omega \times \omega \times \omega$ $= \ \omega^{0'''}$	$\{0 \ldots, \omega \times \omega, \ldots \omega \times \omega \times 0'', \ldots, \omega \times \omega \times 0''', \ldots\ \ldots\}$	$\underbrace{\underbrace{\|\|\ldots\|\|\ldots\ \ldots}_{\omega\text{ times}}\ \underbrace{\|\|\ldots\|\|\ldots\ \ldots}_{\omega\text{ times}}\ \ldots}_{\omega\text{ times}}$
ω^{ω}	$\{0, \ldots, \omega, \ldots, \omega \times 0'', \ldots, \omega \times 0''', \ldots\ \ldots\}$	[see below]

$$\omega^{\omega}: \quad ||\ldots \underbrace{||\ldots||\ldots\ \ldots}_{\omega\text{ times}}\ \underbrace{\underbrace{||\ldots||\ldots\ \ldots}_{\omega\text{ times}}\ \underbrace{||\ldots||\ldots\ \ldots}_{\omega\text{ times}}\ \ldots}_{\omega\text{ times}}\ \underbrace{\underbrace{\underbrace{||\ldots||\ldots\ \ldots}_{\omega\text{ times}}\ \underbrace{||\ldots||\ldots\ \ldots}_{\omega\text{ times}}\ \ldots}_{\omega\text{ times}}\ \underbrace{\underbrace{||\ldots||\ldots\ \ldots}_{\omega\text{ times}}\ \underbrace{||\ldots||\ldots\ \ldots}_{\omega\text{ times}}\ \ldots}_{\omega\text{ times}}\ \ldots\ \ldots}_{\omega\text{ times}}\ \ldots$$

6 A Visualization[‡]



7 Ordinal Precedence v. Cardinal Precedence

We have discussed two different precedence relations, $<_0$ and $<$:

- $<_o$ is the precedence relation for ordinals.

  $\alpha <_o \beta$ means that α precedes β in the hierarchy of ordinals.

[‡] Source: https://commons.wikimedia.org/wiki/File:Omega-exp-omega-labeled.svg. File made available on Wikimedia under the Creative Commons CC0 1.0 Universal Public Domain Dedication. Pop-up casket (talk); original by User:Fool [CC0].

- $<$ is an ordering of set-cardinality.

  $|A| < |B|$ means that there is an injection from A to B (but no bijection).

$\boxed{\textit{Important: } \alpha <_o \beta \text{ does not entail } |\alpha| < |\beta|.}$

8 Ordinals as Blueprints for Large Sets

- An ordinal can be used as a "blueprint" for a sequence of applications of the power set and union operations.
- The farther up an ordinal is in the hierarchy of ordinals, the longer the sequence, and the greater the cardinality of the end result.

Specifically, each ordinal α can be used to characterize the set $\mathfrak{B}_\alpha$:

$$\mathfrak{B}_\alpha = \begin{cases} \mathbb{N}, \text{ if } \alpha = 0 \\ \wp(\mathfrak{B}_\beta), \text{ if } \alpha = \beta' \\ \bigcup\{\mathfrak{B}_\gamma : \gamma <_o \alpha\} \text{ if } \alpha \text{ is a limit ordinal (other than 0)} \end{cases}$$

9 Later Ordinals, Bigger Cardinalities

- By Cantor's Theorem: if $\alpha <_o \beta$, then $|\mathfrak{B}_\alpha| < |\mathfrak{B}_\beta|$.
- For instance:

$$\omega <_o (\omega \times \omega) <_o \omega^\omega <_o {}^\omega\omega. \text{ So: } |\mathfrak{B}_\omega| < |\mathfrak{B}_{\omega\times\omega}| < |\mathfrak{B}_{\omega^\omega}| < |\mathfrak{B}_{{}^\omega\omega}|.$$

10 Initial Ordinals

- **Initial ordinal**: an ordinal that precedes all other ordinals of the same cardinality.
- An initial ordinal κ can be used as proxy for its own cardinality: $\kappa = |\kappa|$.

11 The Beth Hierarchy

- $\beth_\alpha$ (read "beth-alpha") is the initial ordinal of cardinality $|\mathfrak{B}_\alpha|$.
- So: $\beth_\alpha = |\mathfrak{B}_\alpha|$.
- $\beth_0 = |\mathbb{N}|$ and $\beth_{0'} = |\wp(\mathbb{N})|$ (so $\beth_{0'}$ is an **uncountable** ordinal).

Since the beths are *ordinals*, they can be used to define sets bigger than anything we've considered so far. For instance:

- $\mathfrak{B}_{\beth_{0'}}$ (where $\beth_{0'} = |\wp(\mathbb{N})|$)
- $\mathfrak{B}_{\beth_{\beth_\omega}}$ (where $\beth_{\beth_\omega} = |\mathfrak{B}_{\beth_\omega}|$)

12 The Continuum Hypothesis

Continuum Hypothesis There is no set A such that $\beth_0 < |A| < \beth_1$.

Generalized CH There is no set A such that $\beth_\alpha < |A| < \beth_{\alpha+1}$.

13 The Burali-Forti Paradox

Suppose, for *reductio*, that Ω is the set of all ordinals. Then:

- Since Ω consists of every ordinal, it consists of every ordinal that's been introduced so far. But a new ordinal is just the set every ordinal that's been introduced so far. So: Ω **is an ordinal**.

- If Ω was itself an ordinal, it would be a member of itself (and therefore have itself as a predecessor). But no ordinal can be its own predecessor. So: Ω **is not an ordinal**.

So there is no set of all ordinals!