Omega-Sequence Paradoxes

Summary Sheet – 24.118, Spring 2021

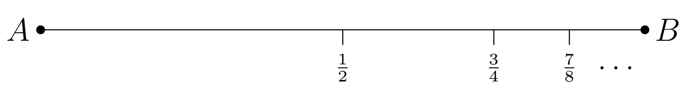
1 What is a Paradox?

A **paradox** is an argument that appears to be valid, and goes from seemingly true premises to a seemingly false conclusion. So we must:

- learn to live with the conclusion;
- learn to live without one of the premises; or
- show that the reasoning is invalid.

An **omega-sequence paradox** is a paradox based on an ω-sequence ($|||\ldots$) or a reverse ω-sequence ($\ldots|||$).

2 Zeno's Paradox[*] [Paradox Grade: 2]

> You wish to walk from point A to point B. In order to do so, you must carry out an ω-sequence of tasks:
>
> $$\begin{array}{ll} \text{Task 1:} & \text{reach } \frac{1}{2} \text{ mark} \\ \text{Task 2:} & \text{reach } \frac{3}{4} \text{ mark} \\ \text{Task 3:} & \text{reach } \frac{7}{8} \text{ mark} \\ \vdots & \vdots \\ \text{Task } n\text{:} & \text{reach } \frac{2^n-1}{2^n} \text{ mark} \\ \vdots & \vdots \end{array}$$
>
> 
>
>
> But it's impossible to complete infinitely many tasks in a finite amount of time. So movement is impossible.

[*]This is a variant of one of several paradoxes attributed to ancient philosopher Zeno of Elea, who lived in the 5th Century BC.

3 Thomson's Lamp[†] [Paradox Grade: 3]

You have a lamp with a toggle button: press the button once and the lamp goes on, press it again and the lamp goes off. Here's what happens:

Time to midnight	Status of lamp shortly thereafter
$60s$	off
$30s$	on
$15s$	off
$7.5s$	on
$\vdots$	$\vdots$
$\frac{60}{2^{2n}}s$	off
$\frac{60}{2^{2n+1}}s$	on
$\vdots$	$\vdots$

Is the lamp on or off at midnight?

- For every time the lamp gets turned off before midnight, there is a later time before midnight when it gets turned on. **So the lamp can't be off at midnight**.
- For every time the lamp gets turned on before midnight, there is a later time before midnight when it gets turned off. **So the lamp can't be on at midnight**.

4 The Demon's Game[‡] [Paradox Grade: 4]

$P_1, P_2, P_3, \ldots$ take turns answering *aye* or *nay*:

- If exactly n people say *aye* ($n \in \mathbb{N}$), each person gets $\$n$.
- If infinitely many people say *aye*, they all get nothing.

It seems rational for P_k to say *aye*: she can't hurt anyone and might help everyone. But if it's rational for P_k it's rational for everyone. So nobody gets anything.

[†]Thomson's Lamp was devised by the late James Thomson, who was a professor of philosophy at MIT (and was married to the great philosopher Judith Jarvis Thomson).

[‡]I learned about this paradox from philosophers Frank Arntzenius, Adam Elga, and John Hawthorne.

5 The Bomber's Paradox[§] [Paradox Grade: 6]

There are infinitely many bombs:

Bomb	When bomb is set to go off
B_0	12:00pm
B_1	11:30am
B_2	11:15am
$\vdots$	$\vdots$
B_k	$\frac{1}{2^k}$ hours after 11:00am
$\vdots$	$\vdots$

Should one of the bombs go off, it will instantaneously disable all other bombs. So a bomb goes off if and only if no bombs have gone off before it:

(0) B_0 goes off $\leftrightarrow$ B_n fails to go off $(n > 0)$.

(1) B_1 goes off $\leftrightarrow$ B_n fails to go off $(n > 1)$.

(2) B_2 goes off $\leftrightarrow$ B_n fails to go off $(n > 2)$.

$\vdots$

($\boldsymbol{k}$) B_k goes off $\leftrightarrow$ B_n fails to go off $(n > k)$.

($\boldsymbol{k+1}$) B_{k+1} goes off $\leftrightarrow$ B_n fails to go off $(n > k+1)$.

$\vdots$

Will any bombs go off?

6 Yablo's Paradox[¶] [Paradox Grade: 8]

There are infinitely many sentences:

Label	Sentence
S_0	"For each $i > 0$, sentence S_i is false"
S_1	"For each $i > 1$, sentence S_i is false"
S_2	"For each $i > 2$, sentence S_i is false"
$\vdots$	$\vdots$
S_k	"For each $i > k$, sentence S_i is false"
$\vdots$	$\vdots$

The meanings of our sentences guarantee that each of the following must be true:

[§] This paradox is due to Josh Parsons, who was a fellow at Oxford until shortly before his untimely death in 2017. (It is a version of Bernadete's Paradox.)

[¶] This paradox was discovered by Steve Yablo, who is a famous philosophy professor at MIT (and was a member of my dissertation committee, many years ago).

(0) S_0 is true $\leftrightarrow$ S_n is false $(n > 0)$.

(1) S_1 is true $\leftrightarrow$ S_n is false $(n > 1)$.

(2) S_2 is true $\leftrightarrow$ S_n is false $(n > 2)$.

$\vdots$

($\boldsymbol{k}$) S_k is true $\leftrightarrow$ S_n is false $(n > k)$.

($\boldsymbol{k+1}$) S_{k+1} is true $\leftrightarrow$ S_n is false $(n > k+1)$.

$\vdots$

Which sentences are true and which ones are false?

7 Bacon's Problem[‖] [Paradox Grade: 7]

- An omega sequence of prisoners: $P_1, P_2, P_3, \ldots$. (P_1 is at the end of the line, in front of her is P_2, in front of him is P_3, and so forth.)
- Each person as assigned a red or blue hat, based on the outcome of a coin toss.
- Everyone can see the hats of people in front of her, but cannot see her own hat (or the hat of anyone behind her).
- At a set time, everyone has to guess the color of her own hat by crying out "Red!" or "Blue!".
- People who correctly call out the color of their own hats will be spared. Everyone else will be shot.

Problem: Find a strategy that $P_1, P_2, P_3, \ldots$ could agree upon in advance and that would guarantee that at most finitely many people are shot.

8 The Three Prisoners[**] [Paradox Grade: 2]

- Three prisoners. Each of them is assigned a red or blue hat, based on the outcome of a coin toss.
- Each of them can see the colors of the others' hats but has no idea about the color of his own hat.
- The prisoners are then taken into separate cells and asked about the color of their hat. They are free to offer an answer or remain silent.

[‖] This paradox is due to USC philosopher Andrew Bacon.

[**] I don't know who invented it, but I learned about it thanks to philosopher and computer scientist Rohit Parikh, from the City University of New York.

Prisoner A	Prisoner B	Prisoner C	Result of following Strategy
red	red	red	Everyone answers incorrectly
red	red	blue	C answers correctly
red	blue	red	B answers correctly
red	blue	blue	A answers correctly
blue	red	red	A answers correctly
blue	red	blue	B answers correctly
blue	blue	red	C answers correctly
blue	blue	blue	Everyone answers incorrectly

Figure 1: The eight possible hat distributions, along with the result of applying the suggested strategy.

- If all three prisoners remain silent, all three will be killed.
- If one of them answers incorrectly, all three will be killed.
- If at least one prisoner offers an answer, and everyone who offers an answer answers correctly, then all three prisoners will be set free.

Problem: Find a strategy that the prisoners could agree upon ahead of time which would guarantee that their chance of survival is above 50%.