Probability

Summary Sheet – 24.118, Spring 2021

1 Two Kinds of Probability

Subjective Probability A person's subjective probability in p is the degree to which she is confident in p.

Example: Jones's subjective probability that it'll rain tomorrow is 0.3 because she is 30% confident that it'll rain tomorrow.

Objective Probability The objective probability of an event is meant to be a feature of the world that does not depend on the beliefs of any particular subject.

Example: the objective probability that a particle of ^{256}Sg will decay in the next 8.9 seconds is 50%.

2 How are they related?

The Objective-Subjective Connection The objective probability of A at time t is the subjective probability that a perfectly rational agent would assign to A, if she had perfect information about the world at times $\leq t$ and no information about the world at times $> t$.*

3 Subjective Probability

A **credence function** for subject S is a function that assigns to each proposition a real number between 0 and 1, representing S's degree of confindence in that proposition

What does it take for a credence function to be rational?

1. internal coherence;
2. update by conditionalization;
3. Bayes' Law;
4. the Principle of Indifference.

*Here I am tacitly presupposing that a perfectly rational agent is always certain about the objective probabilities at t, given full information about how the world is before t. So, in particular, for each complete history of the world up to t, H_t, there is a specification P_t of the objective probabilities at t such that the agent treats H_t and H_tP_t as equivalent. (This assumption is potentially controversial but adds simplicity to our discussion.)

3.1 Internal Coherence

For a credence function to be **internally coherent** is for it to constitute a probability function.

A **probability function**, $p(\ldots)$, is an assignment of real numbers between 0 and 1 to propositions that satisfies the following two coherence conditions:

Necessity $p(A) = 1$ whenever A is a necessary truth

Additivity $p(A \text{ or } B) = p(A) + p(B)$ whenever A and B are incompatible propositions

3.2 Update by Conditionalization

If S is rational, she will update here credences as follows upon learning that B:

$$p^{new}(A) = p^{old}(A|B)$$

where p^{old} is the function describing S's credences before she learned that B, and p^{new} is the function describing her credences after she learned that B.

3.3 Bayes' Law

$p(AB) = p(A) \cdot p(B|A)$

3.4 The Principle of Indifference

Here's what we'd *like* to have in place:

Principle of Indifference Consider a set of propositions and suppose one knows that exactly one of them is true. Suppose, moreover, that one has no more reason to believe any one of them than any other. Then, insofar as one is rational, one should assign equal credence to each proposition in the set.

Unfortunately, this principle leads to inconsistency as stated. For instance:

A factory produces cubes with a side-length $l \leq 1$. What is the probability that $l \in (0, \frac{1}{2}]$?

Argument 1 (length):

- There is just as much reason to think that $l \in (0, \frac{1}{2}]$ as there is to think that $l \in (\frac{1}{2}, 1]$.
- By the Principle of Indifference, $p(l \in (0, \frac{1}{2}]) = p(l \in (\frac{1}{2}, 1])$.
- So $p(l \in (0, \frac{1}{2}]) = \frac{1}{2}$.

Argument 2 (area):

- There is just as much reason to think that $a \in (0, \frac{1}{2}]$ as there is to think that $a \in (\frac{1}{2}, 1]$.
- By the Principle of Indifference, $p(a \in (0, \frac{1}{2}]) = p(a \in (\frac{1}{2}, 1])$.
- So $p(a \in (0, \frac{1}{2}]) = \frac{1}{2}$.

But wait! $l \in (0, \frac{1}{2}] \leftrightarrow a \in (0, \frac{1}{4}]$.

4 Objective Probability

By the Objective-Subjective Connection, our conclusions about rational subjective probability deliver tell us that the objective probabilities:

1. constitute a probability function;
2. update by an analogue of conditionalization;
3. satisfy Bayes' Law;
4. [satisfy a Principle of Indifference?].

5 Yes, but what *is* objective probability?

5.1 Frequentism

What is it for the objective probability of a coin's landing Heads[†] to be 50%?

- According to **frequentism**, it is for 50% of coin tosses to land Heads.
- According to **hypothetical frequentism**, it is for the following subjunctive conditional to be true: if sufficiently many coin tosses took place, 50% of them would land Heads.

5.2 The Law of Large Numbers

Upon reflection, frequentism is obviously incorrect. What is true is this:

> If the coin were tossed a sufficiently large number of times, then it would *with very high probability* land Heads approximately 50% of the time.

More generally and precisely:

Law of Large Numbers Suppose that events of type T have a probability of p of resulting in outcome O. Then, for any real numbers ϵ and δ larger than zero, there is an N such that the following will be true with a probability of at least $1 - \epsilon$:

> If $M > N$ events of type T occur, the proportion of them that result in outcome O will be $p \pm \delta$.

[†] Think of a "coin toss" as the result of observing a particle of ^{256}Sg for 8.9 seconds. If the particle decays within that period, our "coin" is said to have landed Heads; otherwise it is said to have landed Tails.

5.3 Rationalism

- According to **rationalism**, there is nothing more to objective probability than the Objective-Subjective Connection.
- A **localist** agrees with rationalism and adds that the the objective probabilities are only well-defined in certain special circumstances; in particular, circumstances in which there is an unproblematic way of deploying a Principle of Indifference.

6 The Principle of Countable Additivity

(Finite) Additivity $p(A \text{ or } B) = p(A) + p(B)$

whenever A and B are incompatible propositions

Countable Additivity $p(A_1 \text{ or } A_2 \text{ or } \ldots) = p(A_1) + p(A_2) + \ldots$

whenever $A_1, A_2, \ldots$ are countably many propositions with A_i and A_j incompatible for $i \neq j$.

6.1 Against Countable Additivity

- God has selected a positive integer, and that you have no idea which.
- For n a positive integer, what credence should you assign to the proposition, G_n, that God selected n?

Countable Additivity entails that your credences *should remain undefined* (unless you're prepared to give different answers for different choices of n).

Proof: suppose otherwise. Then $p(G_n) = r$, for $r \in [0, 1]$.

- Suppose $r = 0$. By Countable Additivity:

$$\begin{aligned} p(G_1 \text{ or } G_2 \text{ or } G_3 \text{ or } \ldots) &= p(G_1) + p(G_2) + p(G_3) + \ldots \\ &= \underbrace{0 + 0 + 0 + \ldots}_{\text{once for each integer}} \\ &= 0 \end{aligned}$$

- Suppose $r > 0$. By Countable Additivity:

$$\begin{aligned} p(G_1 \text{ or } G_2 \text{ or } G_3 \text{ or } \ldots) &= p(G_1) + p(G_2) + p(G_3) + \ldots \\ &= \underbrace{r + r + r + \ldots}_{\text{once for each integer}} \\ &= \infty \end{aligned}$$

Moral: Countable Additivity entails that there is no way of distributing probability uniformly across a countably infinite set of (mutually exclusive and jointly exhaustive) propositions.

6.1.1 Infinitesimals to the rescue?

What if we had an infinitesimal value ι with the following property?

$$\underbrace{\iota + \iota + \iota + \ldots}_{\text{once for each positive integer}} = 1$$

Then:

$$\begin{aligned} p(G_1 \text{ or } G_3 \text{ or } G_5 \text{ or } \ldots) &= p(G_1) + p(G_3) + p(G_5) + \ldots \\ &= \underbrace{\iota + \iota + \iota + \ldots}_{\text{once for each positive integer}} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} p(G_2 \text{ or } G_4 \text{ or } G_6 \text{ or } \ldots) &= p(G_2) + p(G_4) + p(G_6) + \ldots \\ &= \underbrace{\iota + \iota + \iota + \ldots}_{\text{once for each positive integer}} \\ &= 1 \end{aligned}$$

So, by (finite) Additivity:

$$p(G_1 \text{ or } G_2 \text{ or } G_3 \text{ or } \ldots) = 2 \text{ (!)}$$

6.2 For Countable Additivity

- $X, Y \subseteq \mathbb{Z}^+$
- $p(X)$ is the probability that God selects a number in X.
- $p(X|Y)$ is the probability that God selects a number in X given that She selects a number in Y.

Here is a natural way of characterizing $p(X)$ and $p(X|Y)$:

$$\begin{aligned} p(X|Y) &=_{df} \lim_{n\to\infty} \frac{|X \cap Y \cap \{1, 2, \ldots, n\}|}{|Y \cap \{1, 2, \ldots, n\}|} \\ p(X) &=_{df} p(X|\mathbb{Z}^+) \end{aligned}$$

- $p(X)$ is finitely additive but not countably additive.
- $p(X)$ is not well-defined for arbitrary sets of integers.‡

Also, there is a set S and a partition E_i of $\mathbb{Z}^+$ such that:

- $p(S) = 0$
- $p(S|E_i) \geq 1/2$ for each E_i.

‡For instance, when X consists of the integers k such that $2^m \leq k < 2^{m+1}$, for some even m.

Example:

$S = \{1^2, 2^2, 3^2, \dots\}$; E_i be the set of powers of i (whenever i which is not a power of any other positive integer). In other words:

$$\begin{array}{ccc} S & = & \{1, 4, 9, 16, 25, \dots\} \\ E_1 & = & \{1\} \\ E_2 & = & \{2, 4, 8, 16, 32, \dots\} \\ E_3 & = & \{3, 9, 27, 81, 243, \dots\} \\ \text{[No } E_4\text{, since } 4 = 2^2] & & \\ E_5 & = & \{5, 25, 125, 625, 3125, \dots\} \\ & \vdots & \end{array}$$

6.2.1 Is this really so bad?

Yes. There is a sequence of bets $B_{E_1}, B_{E_2}, B_{E_3}, B_{E_5}, \dots$ such that:

- you are confident that you ought to take each of the bets,
- you are 100% confident that you will lose money if she takes them all.

$\boldsymbol{B_{E_i}}$ **:** Suppose God selects a number in E_i. Then you'll receive \$2 if the selected number is in S and you'll be forced to pay \$1 if the selected number is not in S. (If the selected number is not in E_i, then the bet is called off and no money exchanges hands.)

Problems of this general form are inescapable: *they will occur whenever a probability function on a countable set of possibilities fails to be countably additive.*

7 The Two-Envelope Paradox

- Two envelopes:
  - one contains $\$n$, for n chosen at random from $\mathbb{Z}^+$.
  - the other contains $2n$.
- You are handed one of the envelopes, but don't know which.
- Then you are offered the chance to switch. Should you switch?

An argument for switching:

Say that your envelope contains $\$k$. If k is odd, you should switch. If k is even, there's a 0.5 chance that the other envelope has $\$2k$ and a 0.5 chance that the other envelope has $\$k/2$. So:

$$EV(\text{switch}) = \$k/2 \cdot 0.5 + \$2k \cdot 0.5 = 5/4 \cdot \$k$$

$$EV(\text{not switch}) = \$k$$

7.1 Broome's Variant of the Paradox

- Two envelopes:
  - Toss a die until it lands One or Two. If the die first lands One or Two on the kth toss, place 2^{k-1} in the first envelope.
  - Place twice that amount in the other envelope.