Non-Measurable Sets

Summary Sheet – 24.118, Spring 2021

1 Additive notions of size

- the **length** of two (non-overlapping) line segments placed side by side is the length of the first plus the length of the second;
- the **mass** of two (non-overlapping) objects taken together is the mass of the first plus the mass of the second.
- The **probability** that either of two (incompatible) events occur is the probability that the first occur plus the probability that the second occur;

The notion of **measure** is a very abstract way of thinking about additive notions of size.

2 Generalizing the notion of length

The standard notion of length:

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- $\text{Length}([a, b]) = b - a$.

2.1 The Borel Sets

A **Borel Set** is a set that you can get to by performing finitely many applications of the operations of *complementation* and *countable union* on a family of line segments.*

- The **complementation operation** takes each set A to its complement, $\overline{A} = \mathbb{R} - A$.
- The **countable union operation** takes each countable family of sets $A_1, A_2, A_3, \ldots$ to their union, $\bigcup\{A_1, A_2, A_3 \ldots\}$.

*Formally, the Borel Sets are the members of the smallest set $\mathscr{B}$ such that: (i) every line segment is in $\mathscr{B}$, (ii) if a set is in $\mathscr{B}$, then so is its complement, and (iii) if a countable family of sets is in $\mathscr{B}$, then so is its union.

2.2 Lebesgue Measure

There is exactly one function λ on the Borel Sets that satisfies these three conditions:

Length on Segments $\lambda([a, b]) = b - a$ for every $a, b \in \mathbb{R}$.

Countable Additivity

$$\lambda\left(\bigcup\{A_1, A_2, A_3, \ldots\}\right) = \lambda(A_1) + \lambda(A_2) + \lambda(A_3) + \ldots$$

whenever $A_1, A_2, \ldots$ is a countable family of disjoint sets for each of which λ is defined.

Non-Negativity $\lambda(A)$ is either a non-negative real number or the infinite value ∞, for any set A in the domain of λ.

- a function on the Borel Sets is a **measure** if and only if it satisfies Countable Additivity and Non-Negativity (and assigns the value 0 to the empty set).
- the **Lebesgue Measure** is the (unique) measure λ that satisfies Length on Segments.[†]

3 Uniformity

The Lebesgue Measure, λ, satisfies:

Uniformity $\mu(A^c) = \mu(A)$, whenever $\mu(A)$ is well-defined and A^c is the result of adding $c \in \mathbb{R}$ to each member of A.

3.1 Probability Measures

Two ways of randomly selecting a number from $[0, 1]$:

Standard Coin-Toss Procedure You toss a fair coin once for each natural number. Each time the coin lands Heads you write down a zero, and each time it lands Tails you write down a one. This gives you an infinite binary sequence $\langle d_1, d_2, d_3, \ldots\rangle$, Pick $0.d_1d_2d_3\ldots$ (in binary notation).[‡]

- We get uniformity:

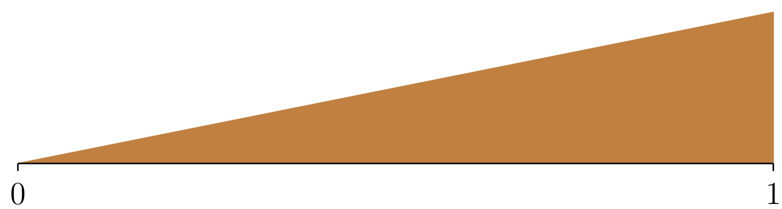
[†]We say that a set $A \subseteq \mathbb{R}$ is **Lebesgue Measurable** if and only if $A = A^B \cup A^0$, for A^B a Borel Set and A^0 a subset of some Borel Set of Lebesgue Measure zero. We apply λ to Lebesgue measurable sets that are not Borel sets by stipulating that $\lambda(A^B \cup A^0) = \lambda(A^B)$.

[‡]Rational numbers have two different binary expansions: one ending in 0s and the other ending in 1s. To simplify the present discussion, I assume that the Coin-Toss Procedure is rerun if the output corresponds to a binary expansion ending in 1s.

- Given certain assumptions about the probabilities of sequences of coin tosses, we get the Lebesgue Measure.

Square Root Coin-Toss Procedure As before, but this time you pick $\sqrt{0.d_1d_2d_3\ldots}$ (in binary notation).

- We do not get uniformity:



4 Non-Measurable Sets

- There are subsets of $\mathbb{R}$ that are **non-measurable**:

  They cannot be assigned a measure by any extension of λ, without giving up on Non-Negativity, Countable Additivity, or Uniformity.

5 The Axiom of Choice

Proving that there are non-measurable sets requires:

Axiom of Choice Every set of non-empty, non-overlapping sets has a choice set.

(A **choice set** for set A is a set that contains exactly one member from each member of A.)

6 Defining the Vitali Sets

6.1 A sketch of the construction

- Define an (uncountable) partition $\mathcal{U}$ of $[0, 1)$.
- Use the Axiom of Choice to pick a representative from each cell of $\mathcal{U}$.
- Use these representatives to define a (countable) partition $\mathcal{C}$ of $[0, 1)$.
- A Vitali Set is a cell of $\mathcal{C}$.

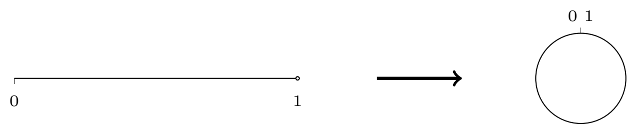
6.2 Defining $\mathcal{U}$

$a, b \in [0, 1)$ are in the same cell if and only if $a - b \in \mathbb{Q}$.

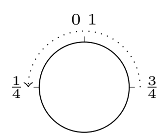
6.3 Defining $\mathcal{C}$

- $\mathcal{C}$ has a cell C_q for each rational number $q \in \mathbb{Q}^{[0,1)}$.
- C_0 is the set of representatives of cells of $\mathcal{U}$.
- C_q is the set of numbers $x \in [0, 1)$ which are at a "distance" of q from the representative of their cell in $\mathcal{U}$.

Here "distance" is measured by bending $[0, 1)$ into a circle:



and traveling counter-clockwise. For instance, $\frac{1}{4}$ is at "distance" $\frac{1}{2}$ from $\frac{3}{4}$:



7 A Vitali Set Cannot Be Measured

7.1 Assumptions

Countable Additivity

$$\lambda\left(\bigcup\{A_1, A_2, A_3, \ldots\}\right) = \lambda(A_1) + \lambda(A_2) + \lambda(A_3) + \ldots$$

whenever $A_1, A_2, \ldots$ is a countable family of disjoint sets for each of which λ is defined.

Non-Negativity $\lambda(A)$ is either a non-negative real number or the infinite value ∞, for any set A in the domain of λ.

Uniformity $\mu(A^c) = \mu(A)$, whenever $\mu(A)$ is well-defined and A^c is the result of adding $c \in \mathbb{R}$ to each member of A.

7.2 The Proof

- Suppose otherwise: $\lambda(C_q)$ is well-defined for some $q \in \mathbb{Q}^{[0,1)}$.
- By Uniformity, $\lambda(C'_q) = \lambda(C_q)$ for any $q' \in \mathbb{Q}^{[0,1)}$.
- By Non-Negativity, $\lambda(C_q)$ is either 0, or a positive real number, or ∞.
- By Countable Additivity, it can't be any of these:
  - Suppose $\lambda(C_q) = 0$. By Countable Additivity:

$$\begin{aligned} \lambda([0,1)) &= \lambda(C_q) + \lambda(C_{q'}) + \ldots \\ &= \underbrace{0 + 0 + 0 + \ldots}_{\text{once for each integer}} \\ &= 0 \end{aligned}$$

  - Suppose $\lambda(C_q) = r > 0$. By Countable Additivity:

$$\begin{aligned} \lambda([0,1)) &= \lambda(C_q) + \lambda(C_{q'}) + \ldots \\ &= \underbrace{r + r + r + \ldots}_{\text{once for each integer}} \\ &= \infty \end{aligned}$$

Moral: There is no way of assigning a measure to a Vitali set without giving up on Uniformity, Non-Negativity or Countable Additivity.

8 Revising Our Assumptions?

- Giving up on **Uniformity** means *changing the subject*: the whole point of our enterprise is to find a way of extending the notion of Lebesgue Measure without giving up on uniformity.
- **Non-Negativity** and **Countable Additivity** are not actually needed to prove the existence of non-measurable sets.
- Some mathematical theories would be seriously weakened by giving up on the **Axiom of Choice**.