Computability

Summary Sheet – 24.118, Spring 2021

1 The Main Result

- We'll focus on functions $f : \mathbb{N} \to \mathbb{N}$.
- For a computer program to **compute** f is for it to yield $f(n)$ as output whenever it is given n as input ($n \in \mathbb{N}$).
- *Theorem:* not every function is computable.

  (And I can give you examples!)

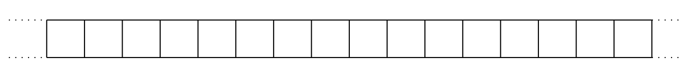
2 How We'll Get There

- Turing Machines are computers of an especially simple sort.
- We'll see that some functions are not Turing-computable.
- But: any function that can be computed using an ordinary computer is also computed by some Turing Machine.

3 Turing Machines

Hardware

Memory tape A long strip of paper, divided into cells:



(An assistant is ready to add paper on either end, as needed.)

Tape-reader At any given time, the reader is positioned on a cell of the memory tape and is able to perform any of the following functions:

- Read the symbol written on the cell
- Write a new symbol on the cell
- Move one cell to the left
- Move one cell to the right

Software

A finite list of **command lines**:

$$\langle\text{current state}\rangle\ \langle\text{current symbol}\rangle\ \langle\text{new symbol}\rangle\ \langle\text{direction}\rangle\ \langle\text{new state}\rangle$$

Think of a command line as encoding the following instruction:

> If you are in ⟨current state⟩ and your reader sees ⟨current symbol⟩ written on the memory tape, replace ⟨current symbol⟩ with ⟨new symbol⟩. Then move one step in direction ⟨direction⟩, and go to ⟨new state⟩.

Operation

- Start out in state 0. Then carry out the following procedure, for as long as you are able:
  - Perform the instruction corresponding to the (first) command line that matches your current state and the symbol on which your reader is positioned.
  - Repeat.
- If you are unable carry out the procedure, halt.

4 A Turing Machine Simulator

`http://morphett.info/turing/`

5 Computing functions on a Turing Machine

Computability:

- For a computer program to **compute** f is for it to yield $f(n)$ as output whenever it is given n as input.

Turing Computabiity:

- M takes n ($n \in \mathbb{N}$) as **input** if it starts out with a tape that contains only a sequence of n ones (with the reader positioned at the left-most one, if $n > 0$).
- M delivers $f(n)$ as **output** if it halts with a tape that contains only a sequence of $f(n)$ ones (with the reader positioned at the left-most one, if $n > 0$).
- M **computes** a function $f(x)$ if and only if it delivers $f(n)$ as output whenever it is given n as input.

6 The Main Result

- We'll focus on functions $f : \mathbb{N} \to \mathbb{N}$.
- For a computer program to **compute** f is for it to yield $f(n)$ as output whenever it is given n as input ($n \in \mathbb{N}$).
- *Theorem:* not every function is computable.

  (And I can give you examples!)

6.1 The Overall Plan

- Turing Machines are computers of an especially simple sort.
- We'll see that some functions are not Turing-computable.
- But: any function that can be computed using an ordinary computer is also computed by some Turing Machine.

6.2 Computing functions on a Turing Machine

- Simplifying Assumptions:
  - We'll focus on *one symbol* Turing Machines (where the only admissible symbols are ones and blanks).
  - We'll assume that the tape is only unbounded on the right.
- Turing Computabiity:
  - M **computes** a function $f(x)$ if and only if it delivers $f(n)$ as output whenever it is given n as input.
  - M takes n ($n \in \mathbb{N}$) as **input** if it starts out with a tape that contains only a sequence of n ones (with the reader positioned at the left-most one, if $n > 0$).
  - M delivers $f(n)$ as **output** if it halts with a tape that contains only a sequence of $f(n)$ ones (with the reader positioned at the left-most one, if $n > 0$).

6.3 Coding Turing Machines as Numbers

The Plan

Turing Machine $\to$ Sequence of symbols $\to$ Sequence of numbers $\to$ Unique number

Sequence of symbols → Sequence of numbers

State Symbols:	Tape Symbols:	Movement Symbols:
"0" → 0	"_" → 0	"r" → 0
"1" → 1	"1" → 1	"*" → 1
⋮		"l" → 2

Sequence of numbers → Unique number

Codes the sequence $\langle n_1, n_2, \ldots, n_k \rangle$ as the number:

$$p_1^{n_1+1} \cdot p_2^{n_2+1} \cdot \ldots \cdot p_k^{n_k+1}$$

where p_i is the ith prime number.

(Treat any number that doesn't code a valid sequence of command lines as a code for the "empty" Turing Machine.)

6.4 An example

$$\begin{array}{c} 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \\ \downarrow \\ 2^{0+1} \cdot 3^{0+1} \cdot 5^{0+1} \cdot 7^{0+1} \cdot 11^{0+1} \\ \downarrow \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ \downarrow \\ 0 \quad \_ \quad \_ \quad r \quad 0 \end{array}$$

6.5 The Halting Function

- $H(n, m) = \begin{cases} 1 & \text{if the } n\text{th Turning Machine halts when given input } m; \\ 0 & \text{otherwise.} \end{cases}$

  For instance: $H(2310, 0) = 0$ and $H(2310, 2310) = 1$.

- $H(n) = H(n, n)$

  For instance: $H(2310) = 1$.

6.5.1 $H(n)$ is not Turing-computable

- Assume for *reductio*: Turing Machine M^H computes $H(n)$.
- Construct Turing Machine M^I, which behaves as follows on input k:

  Step 1: Check whether $H(k)$ (using M^H).

  Step 2: $\begin{cases} \text{If } H(k) = 1, \text{ go right forever.} \\ \text{If } H(k) = 0, \text{ halt.} \end{cases}$

- *Informally:* What happens when you run M^I on input $\overline{M^I}$? It figures out whether it itself would halt on input $\overline{M^I}$. If the answer is yes, it goes off on an infinite task; if the answer is no, it immediately halts.
- *Formally:* $H(\overline{M^I})$ 1 or 0?
  - Suppose $H(\overline{M^I}) = 1$. Then (by Step 2) M^I goes right forever on input $\overline{M^I}$. So $H(\overline{M^I}) = 0$.
  - Suppose $H(\overline{M^I}) = 0$. Then (by Step 2) M^I halts on input $\overline{M^I}$. So $H(\overline{M^I}) = 1$.
- So M^I is impossible. So M^H isn't computable after all.

6.6 The Busy Beaver Function

- **Productivity(M)** $= \begin{cases} k, & \text{if } M \text{ yields output } k \text{ on an empty input} \\ 0, & \text{otherwise} \end{cases}$
- $\boldsymbol{BB(n)} =$ the productivity of the most productive (one-symbol) Turing Machine with n states or fewer.

6.6.1 $BB(n)$ is not Turing-computable

- Assume for *reductio*: Turing Machine M^{BB} computes $BB(n)$.
- Construct Turing Machine M^I, which behaves as follows on an empty input:

  Step 1: Print a sequence of k ones, for a certain k (specified below).
  Result: k.

  Step 2: Duplicate your string of ones.
  Result: $2k$.

  Step 3 Apply BB to your string of ones (using M^{BB}).
  Result: $BB(2k)$.

  Step 4 Add one to your string of ones.
  Result: $BB(2k) + 1$.

- Let $k = b + c + d$

  $b =$ the number of states used in Step 2 (to duplicate)

  $c =$ the number of states used in Step 3 (to apply BB)

  $d =$ the number of states used in Step 4 (to add one)

  Note: since a Turing Machine can output k using k states,

$$\overline{M^I} = k + b + c + d = 2k$$

- M^{BB} is impossible:
  - At Stage 3, it produces as long a sequence of ones as a machine with $2k$ states could possibly produce.
  - But (as noted above) $\overline{M^I} = 2k$.
  - So at Stage 3, it produces as long a sequence of ones as it itself could possibly produce.
  - So at Stage 4, it produces a *longer* string of ones than it itself could possibly produce.
- So M^H isn't computable after all.

7 The Universal Turing Machine

There is a **Universal Turing Machine**, M^U, which does the following:

- if the mth Turing Machine halts given input n, leaving the tape in configuration p, then M^U halts given input $\langle m, n\rangle$ leaving the tape in configuration p.
- if the mth Turing Machine never halts given input n, then M^U never halts given input $\langle m, n\rangle$.

8 The Fundamental Theorem

The reason Turing Machines are so valuable is that it is possible to prove the following theorem:

Fundamental Theorem of Turing Machines A function from natural numbers to natural numbers is Turing-computable if and only if it can be computed by an ordinary computer, assuming unlimited memory and running time.

- One shows that every Turing-computable function is computable by an ordinary computer (given unlimited memory and running time) by showing that one can program an ordinary computer to simulate any given Turing Machine.

- One shows that every function computable by an ordinary computer (given unlimited memory and running time) is Turing-computable by showing that one can find a Turing Machine that simulates any given ordinary computer.

9 Church-Turing

Computer scientists tend to think that something stronger than the Fundamental Theorem is true:

Church-Turing Thesis A function is Turing-computable if and only if it can be computed algorithmically.

For a problem to be solvable **algorithmically** is for it to be possible to specify a finite list of instructions for solving the problem such that:

1. Following the instructions is guaranteed to yield a solution to the problem, in a finite amount of time.
2. The instructions are specified in such a way that carrying them out requires no ingenuity of any kind: they can be followed mechanistically.
3. No special resources are required to carry out the instructions: they could in principle be carried out by a machine built from transistors.
4. No special physical conditions are required for the computation to succeed (no need for faster-than-light travel, special solutions to Einstein's equations, etc).